

Statistical and Low Temperature Physics (PHYS393)

5. Dilution Cooling

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The dilution refrigerator

Cooling by evaporation of helium-4 liquid can only reach about 1.3 K. Below this temperature, the vapour pressure is very small, so that very little would evaporate.

It is possible to overcome this limitation using a mixture of liquid helium-3 and liquid helium-4.

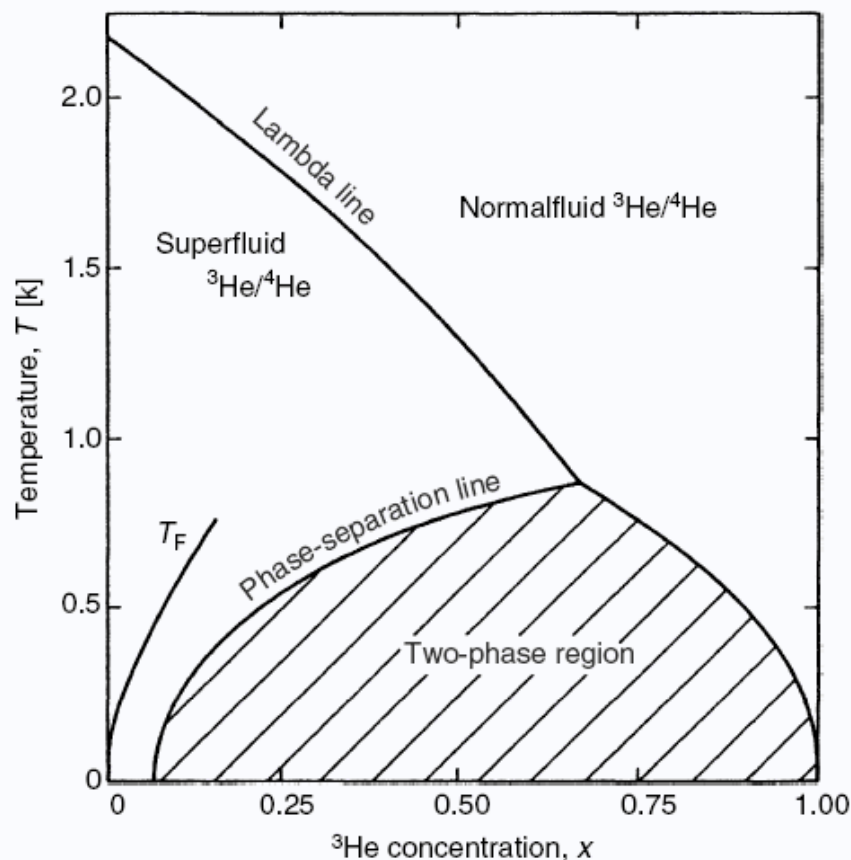
The way is to "evaporate" pure liquid helium-3 into the mixture.

This is done in the dilution refrigerator. Using this method, it is possible to reach into the milliKelvin range.

Properties of the liquid ^3He - ^4He mixtures

Liquid ^3He - ^4He Mixtures

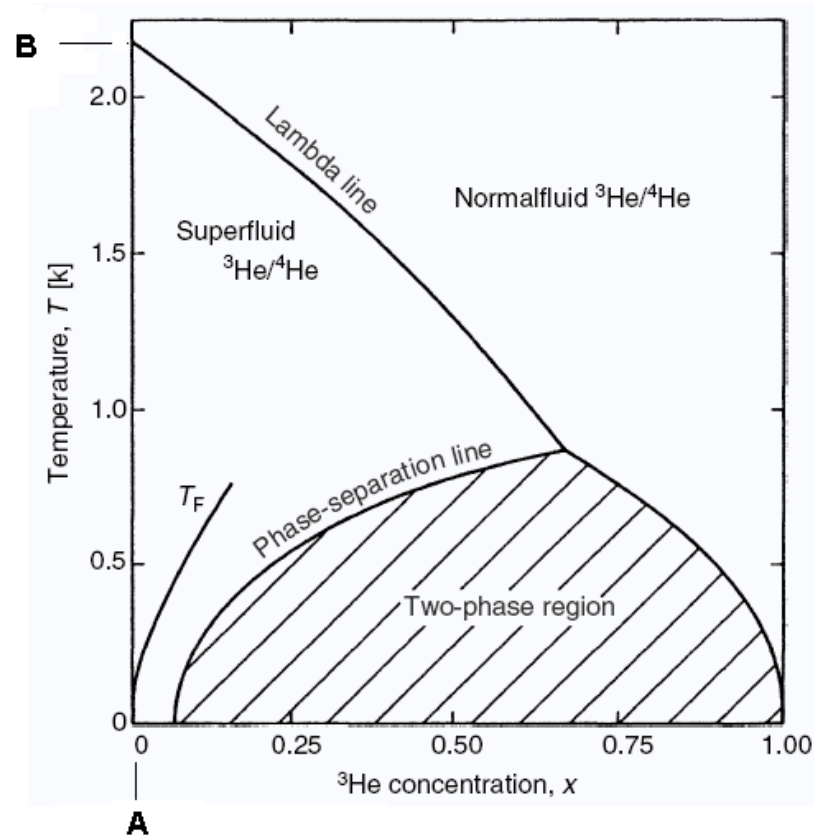
To understand the dilution refrigerator, we start with the phase diagram. This tells us how the mixture behaves at different concentrations and temperatures.



D.S. Betts: An Introduction to Millikelvin Technology (Cambridge University Press, 1989)
J. Wilks, D.S. Betts: An Introduction to Liquid Helium, 2nd edn. (Clarendon, Oxford 1987)

The Phase Diagram

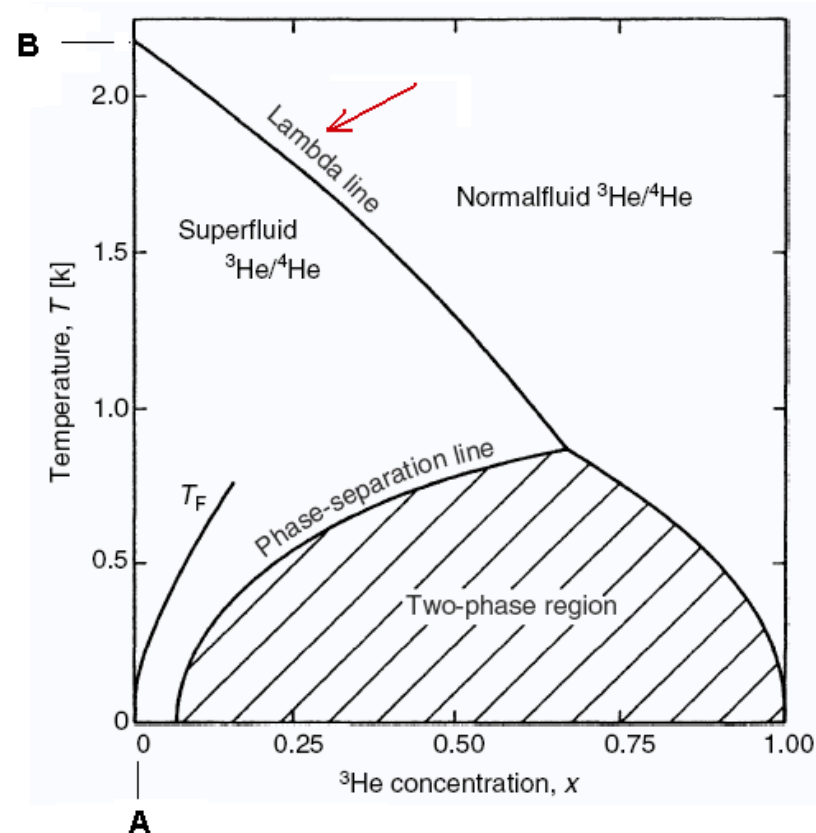
Let x be the fraction of ^3He atoms in the ^3He - ^4He mixture. In the figure, x is the horizontal axis.



At point A: $x = 0$ means pure ^4He .

The Phase Diagram

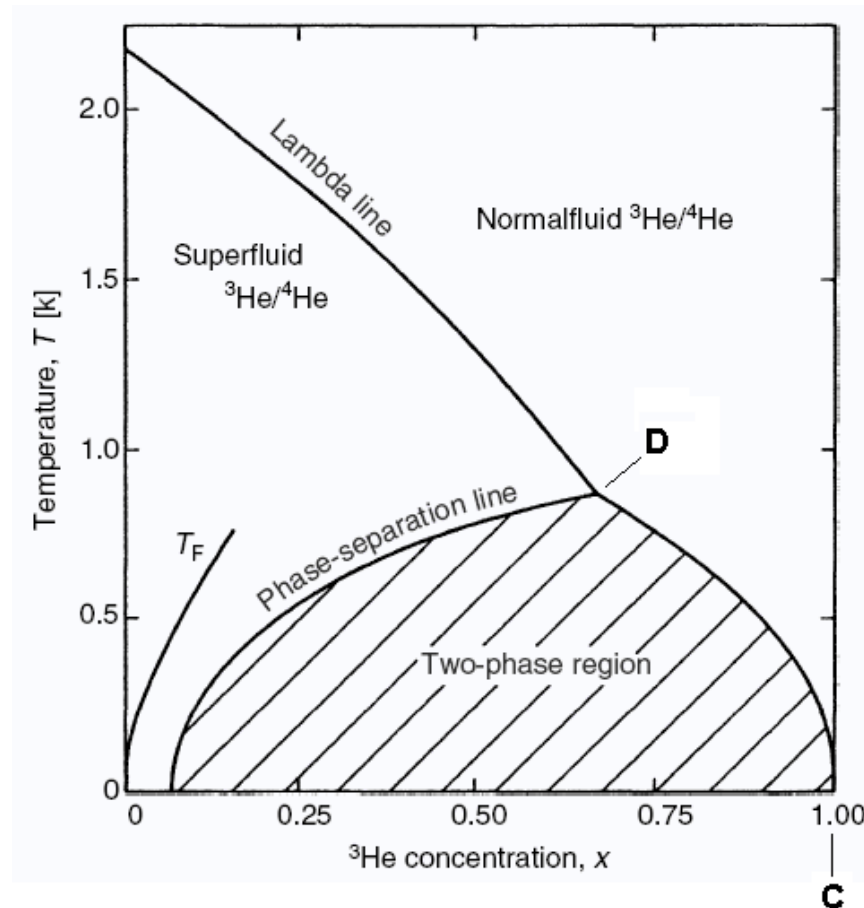
The Lambda line indicates the temperature at which superfluid transition takes place for a given x .



At point B: At $x = 0$, the Lambda line shows that superfluid transition takes place at the temperature $T = 2.2$ K.

The Phase Diagram

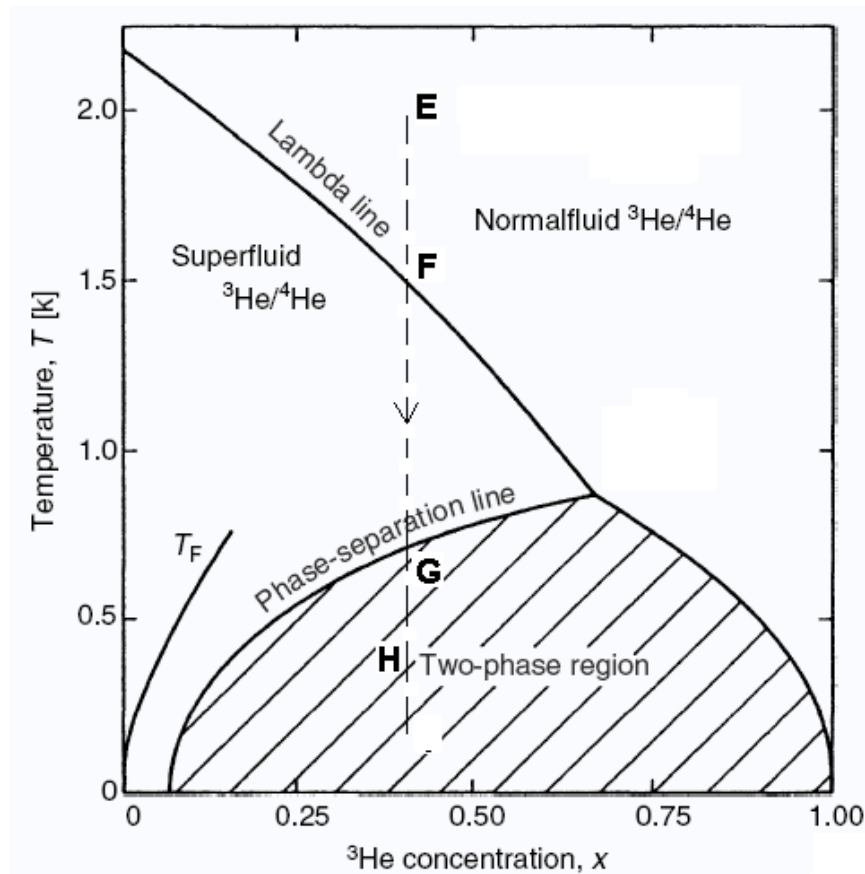
At point C: $x = 1$ is pure ^3He .



At point D: The Lambda line stops at the shaded region. So it is not clear if pure ^3He (at C) can become superfluid. In fact it can, at the much lower temperature of 2.5 mK.

The Phase Diagram

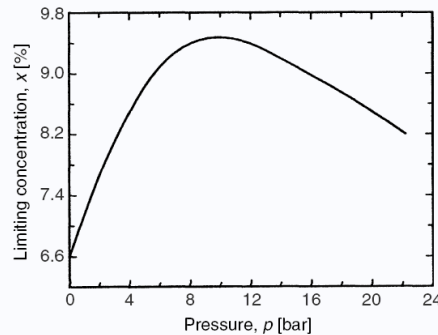
Along the dashed line: Suppose we start with a mixture at a concentration and temperature at point E. When this is cooled down to the temperature at point F, it would change to a superfluid.



If it is cooled further, it would reach the shaded region at G. What if it is cooled below that to a temperature at point H?

The Phase Diagram

The shaded region is meant to indicate that it is not possible for the mixture to exist at a concentration and temperature in that region.

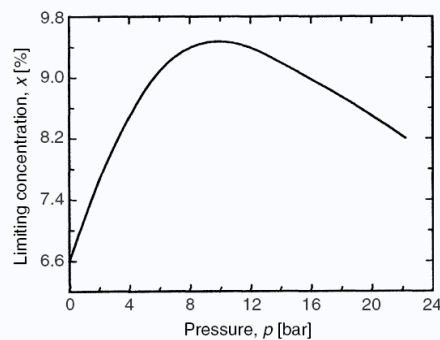


Limiting low-temperature concentration of ^3He in ^4He at $T = 50\text{ mK}$

For a temperature at point H, it is only possible to have concentrations that are either smaller than that at point J, or higher than at point K.

The Phase Diagram

So if a mixture is cooled down to a temperature at H, it would separate into two layers.



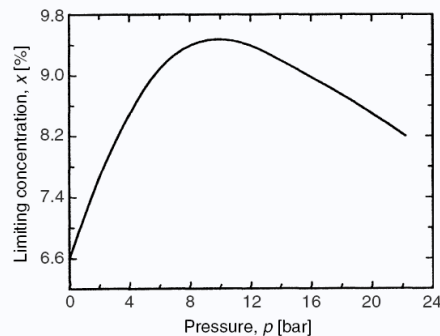
Limiting low-temperature concentration of ^3He in ^4He at $T = 50$ mK

One layer has the concentration at J, and the other layer has the concentration at K.

The Phase Diagram

At J, the concentration x of ^3He is lower, and the mixture is called the dilute phase.

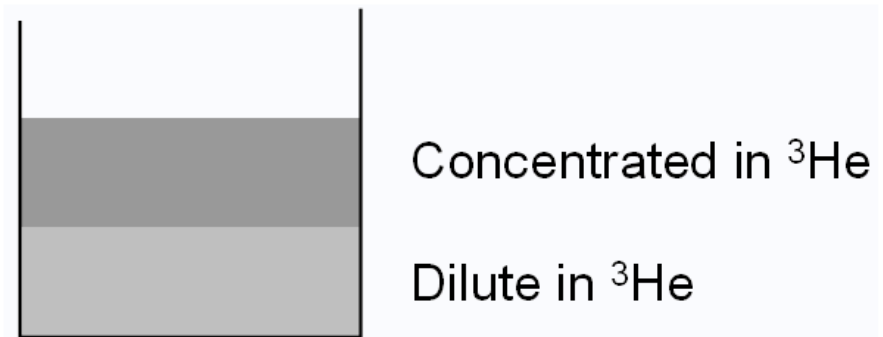
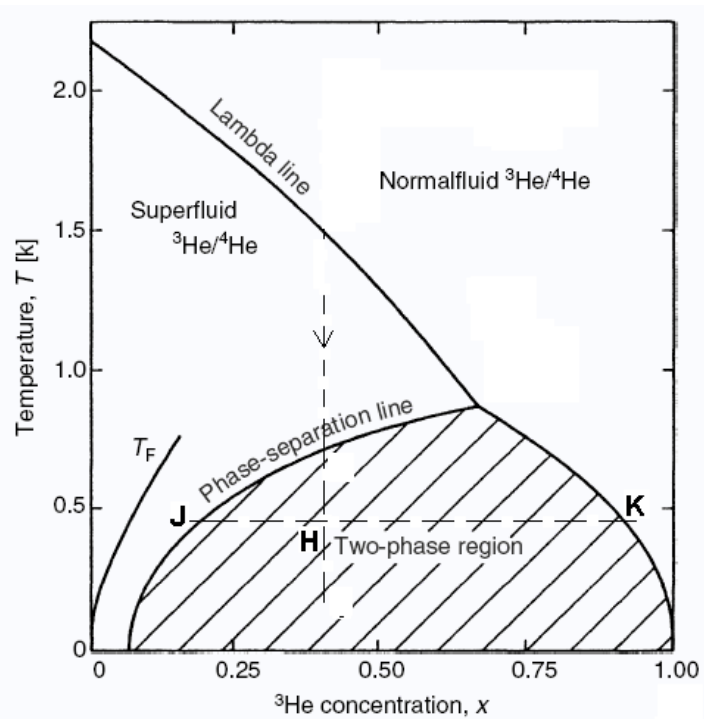
At K, the concentration is higher, and it is called the concentrated phase.



Limiting low-temperature concentration of ^3He in ^4He at $T = 50\text{ mK}$

The Phase Diagram

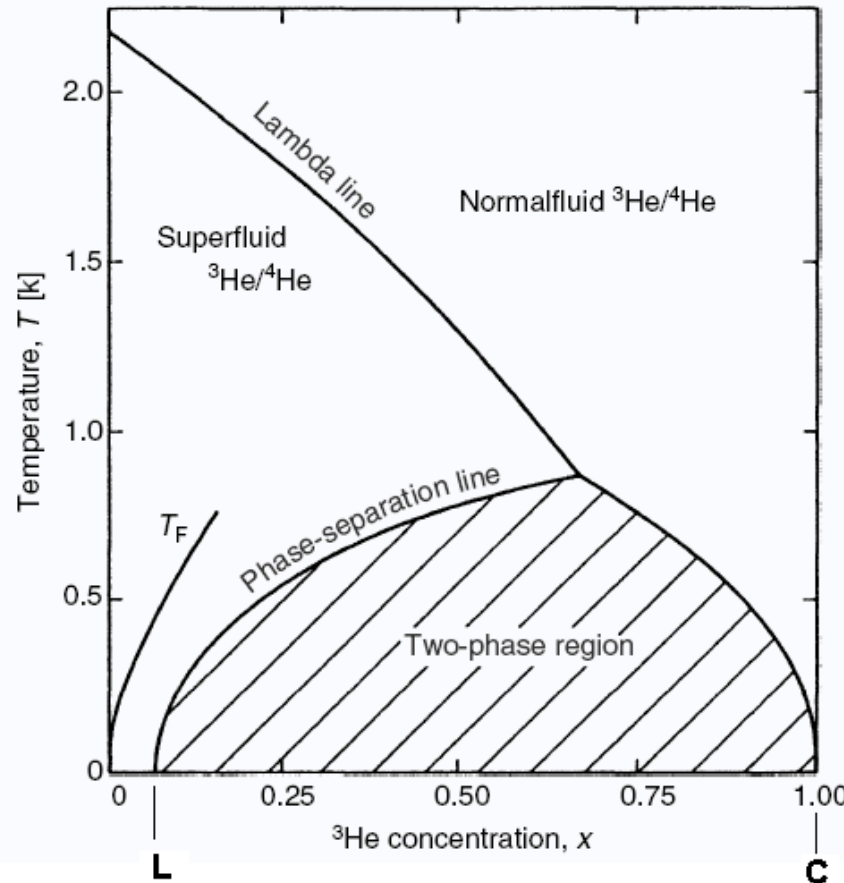
Because ^3He is less dense than ^4He , the concentrated phase floats on top of the dilute phase.



Note that the dilute phase is superfluid, whereas the concentrated phase is normal fluid.

The Phase Diagram

At point L: When we reach a temperature below 0.1 K, the highest concentration possible is $x = 6.6\%$.

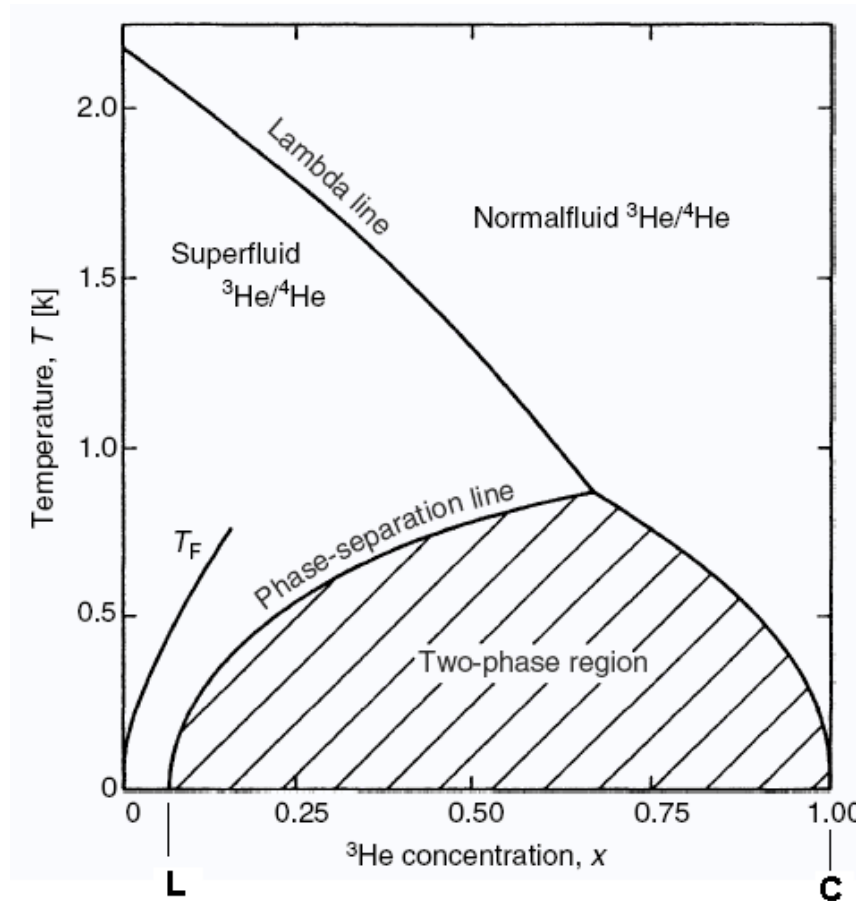


If we start at a higher temperature and concentration and cool down below 0.1 K, it would always separate into 2 layers:

1. One layer has 6.6% concentration in ^3He - at L.
2. The other is nearly pure ^3He - at C.

The Phase Diagram

At point L: Note that in the dilute phase, the solubility of ^3He would remain 6.6% even as the temperature approaches 0 K.



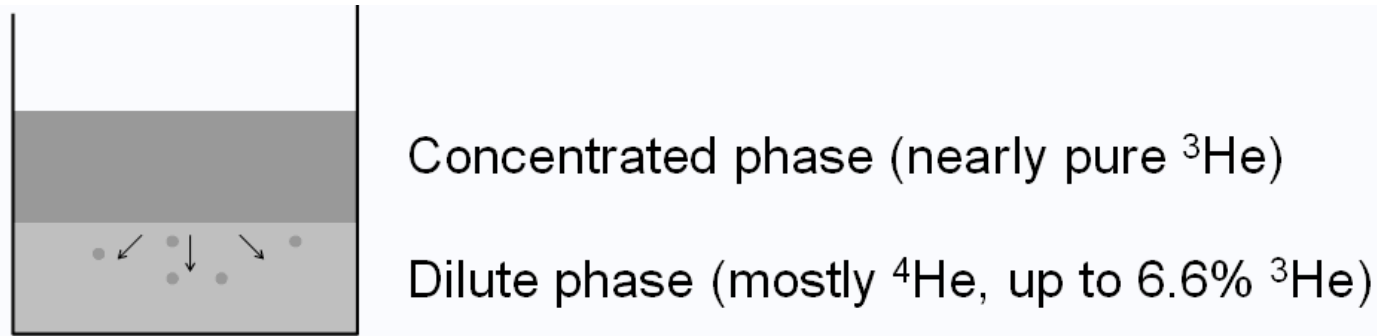
This is the property that makes the dilution refrigerator possible.

Cooling by dilution

Cooling by dilution

Suppose that we start with a layer of pure helium-3 floating on a layer of pure helium-4 at 0.1 K.

Helium-3 would diffuse into the helium-4 layer below. The reverse would not happen, as we know from the phase diagram.



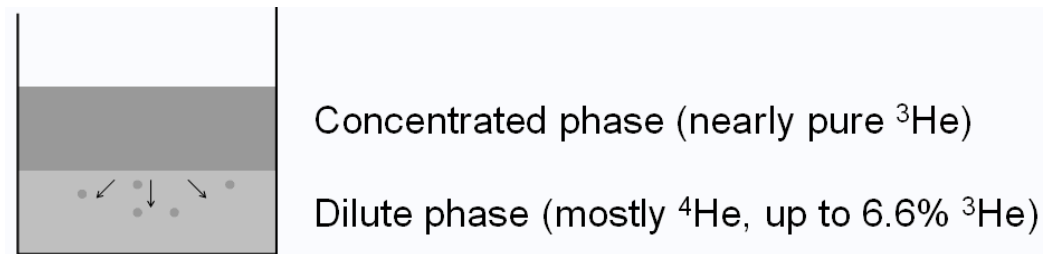
When this happens, the temperature would fall.

We may compare this to evaporation. Instead of vaporising to the vacuum above, the helium-3 "vaporises" into the liquid below.

In this case, it is mixing instead of evaporation.

Cooling by dilution

By mixing into the lower layer, the helium-3 above is effectively being diluted. Hence the term "dilution cooling."



This continues, and the concentration of helium-3 in the in the bottom layer increases until it reaches 6.6%. Then the mixing stops.

In order to continue cooling, we must somehow remove the helium-3 dissolved in the dilute phase. This also has an analogy with cooling by evaporation, where we have to pump out the vapour to prevent it from being saturated.

We have previously seen that the vapour pressure of helium falls exponentially with decreasing temperature.

The vapour pressure, P , is directly related to the rate at which the helium atoms vaporise from the liquid. Therefore it is also directly related to the rate of removing heat from the liquid:

$$\dot{Q} \propto P \propto e^{-1/T}$$

\dot{Q} is also called the cooling power, and is used to compare the performance of different refrigerators.

So the cooling power of the evaporation refrigerator falls exponentially with decreasing temperature.

In dilution cooling, the helium-3 concentration, x , in the dilute phase is directly related to the rate at which helium-3 leaves the concentrated phase.

To determine the rate at which heat is removed, we must consider the heat change of mixing, ΔH . As we shall see later, this heat change is proportional to T^2 .

The cooling power of helium-3 dilution is therefore

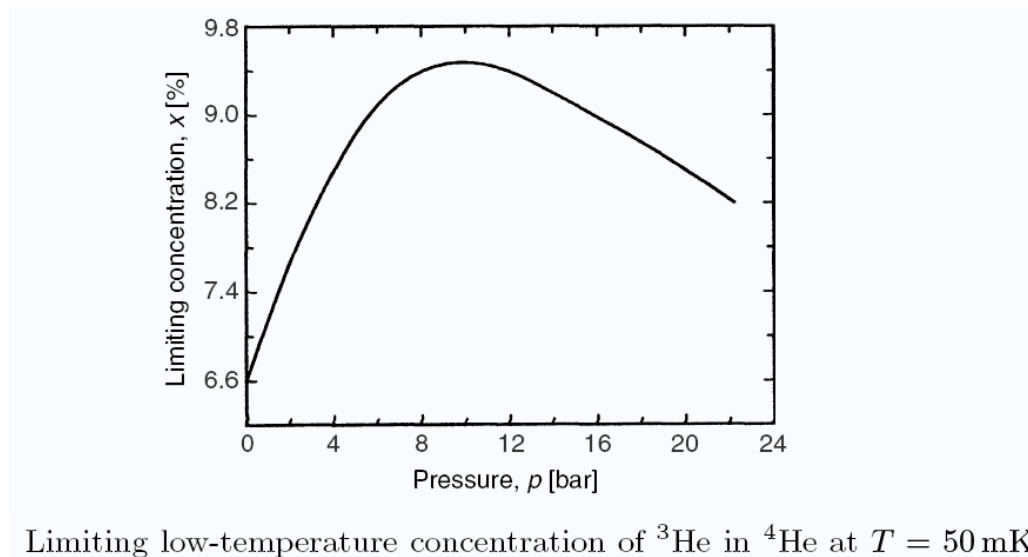
$$\dot{Q} \propto x\Delta H \propto T^2$$

Cooling power

We have seen that the cooling power

$$\dot{Q} \propto x \Delta H$$

depends on the concentration x .

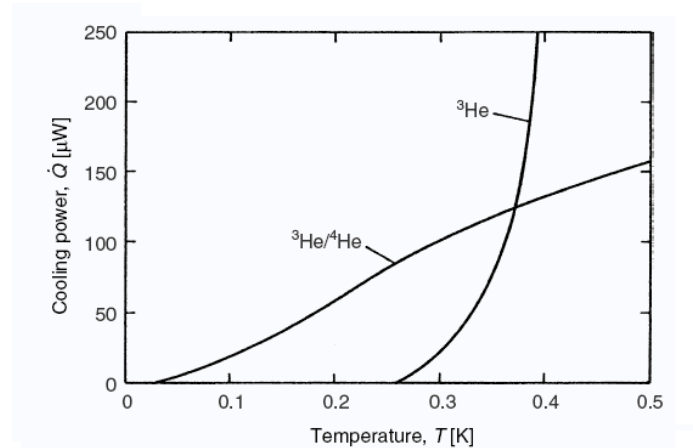


G.E. Watson, et al: Phys. Rev. 188, 384 (1969)

This figure shows that it is possible to increase the limiting concentration above 6.6% by increasing the pressure. This offers one way of increasing the cooling power.

Cooling power

The figure compares the cooling power of helium-3 evaporation, and ^3He - ^4He dilution, for the same helium-3 circulation rate.



assuming the same helium gas circulation rate of 5 l s^{-1}

O.V. Lounasmaa: Experimental Principles and Methods Below 1K (1974)

From previous reasonings, we can understand why the cooling power for the evaporation falls much faster than that for dilution.

Below 0.3 K, the dilution refrigerator clearly has a much higher cooling power than the evaporation refrigerator.

^3He - ^4He mixtures as Fermi liquids

It is possible to derive the heat change of mixing. This would be useful for estimating the cooling power.

Helium-3 has nuclear spin $I = 1/2$. It is a fermion. It obeys the Fermi-Dirac statistics. So we hope that the same formula for the heat capacity of electrons can be used for the ^3He - ^4He mixture:

$$C_3 = \frac{\pi^2 T}{2 T_F} R \text{ at } T \ll T_F$$

where C_3 denotes the heat capacity of helium-3.

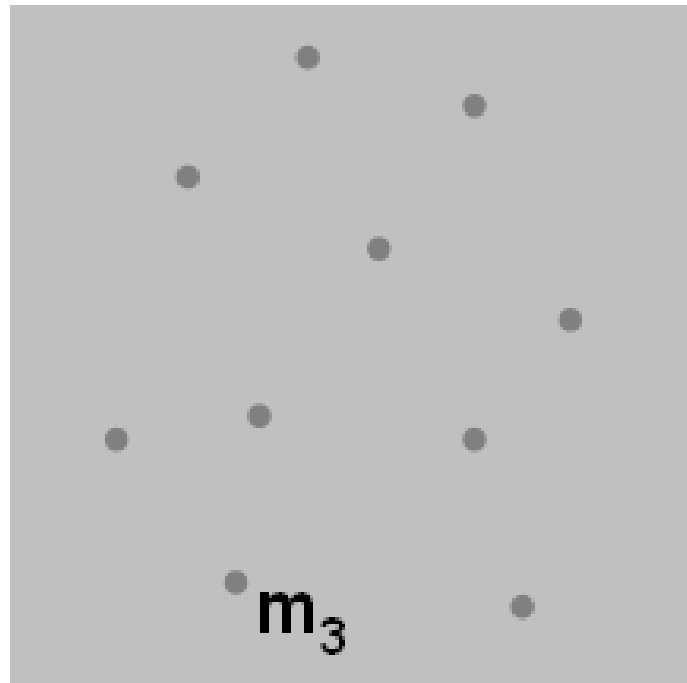
From the lectures on electrons in metal, we have seen that the Fermi temperature $T_F = E_F/k_B$, and the Fermi energy is

$$E_F = \frac{\hbar^2}{2m_3} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$$

where m_3 denotes the mass of the helium-3 atom.

^3He - ^4He mixtures as Fermi liquids

The problem with the formulae for electrons is that they are derived assuming that the particles do not interact with one another.



This assumption is not valid here. In the dilute phase, the helium-3 atoms are very close to the helium-4. In the concentrated phase, the helium-3 are very close to themselves.

^3He - ^4He mixtures as Fermi liquids

It turns out that the helium-3 atoms in the dilute phase can be approximated as heavier particles in a vacuum.



For example, at 6.6% concentration, if we use a value of $2.45m_3$ for the mass of helium-3 instead of the actual m_3 , the formulae would still give a reasonable answer.

The higher mass is called the effective mass, and often denoted by m^* . This has been demonstrated experimentally in 1966.

A. C. Anderson, et al, Physical Review Letters, vol. 16 (1966), pp. 263-264

(For pure helium-3, $m^* = 2.78m_3$).

Another point to note for the formula

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R \text{ at}$$

is the condition that $T \ll T_F$.

For electrons in metal, we have seen that the Fermi energy is much higher than $k_B T$ at room temperature.

What is the Fermi temperature for helium-3 in the dilute phase? Is it still higher than the temperature we are interested in?

At 6.6% concentration, and using the effective mass of $m^* = 2.45m_3$, we would find using the formulae that T_F is about 1 K.

The dilution refrigerator typically operates below 0.1 K. This should be well within the valid range for the Fermi gas formulae.

Heat change of mixing

Heat change of mixing

To derive the heat change of mixing, we need a few ideas and formulae from thermodynamics. Since the movement of particles from concentrated to dilute phase is essentially a change in phase, we need the condition for phase equilibrium:

$$\mu_C = \mu_D$$

where μ is the chemical potential, subscript C is for concentrated phase, and D for the dilute phase.

The chemical potential is given by

$$\mu = H - TS$$

where H is the enthalpy per mole, and S the entropy per mole of the phase.

I shall start with a quick summary on the basic physics behind the equilibrium condition.

Heat change of mixing

The enthalpy H is given by

$$H = U + pV,$$

where U is the internal energy, p the pressure and V the volume. So

$$\mu = H - TS = U + pV - TS.$$

The equilibrium equation is a statement that the change in chemical potential if one phase is changed to the other, is zero. This may also be expressed as:

$$\Delta\mu = \Delta U + p\Delta V - T\Delta S = 0.$$

This assumes that pressure and temperature are the same in both phases.

There would always be a pressure and a temperature gradient in the refrigerator, since the concentrated phase is being cooled and it is on top. But since the volumes are small and the two phases are in close contact, we shall assume that it is approximately true.

Heat change of mixing

The equilibrium condition is:

$$\Delta U + p\Delta V - T\Delta S = 0.$$

To understand this physically, note that for a reversible change, $T\Delta S = \Delta Q$, the heat input. $p\Delta V$ is the work done by one phase if it expands on changing to the other phase.

So the left hand side is a statement that the total energy of one mole of a phase remains the same, when it changes into another phase.

If this is the case, then the two phases would remain in equilibrium.

If, on the other hand, there is a net release in energy when one phase changes into the other, then this change would take place, and there would be no equilibrium.

Returning to the equilibrium condition

$$\mu_C = \mu_D,$$

since $\mu = H - TS$, we may write this as

$$H_C - TS_C = H_D - TS_D$$

Remember that subscript C is for concentrated phase, and D for the dilute phase. The enthalpy change of mixing is therefore:

$$H_D - H_C = TS_D - TS_C$$

This is the heat change of mixing that we want to estimate.

To do so, we need to find the entropies S_C and S_D in both phases.

Heat change of mixing

What we need the entropy in each phase. This can be determined from the specific heat capacity as follows.

Conservation of energy, or the first law of thermodynamics, tells us that:

$$dQ = dU + dW$$

dW is the mechanical work done - by expansion or contraction. Since we have a liquid, the volume change is very small. If we neglect this, we have

$$dQ = dU$$

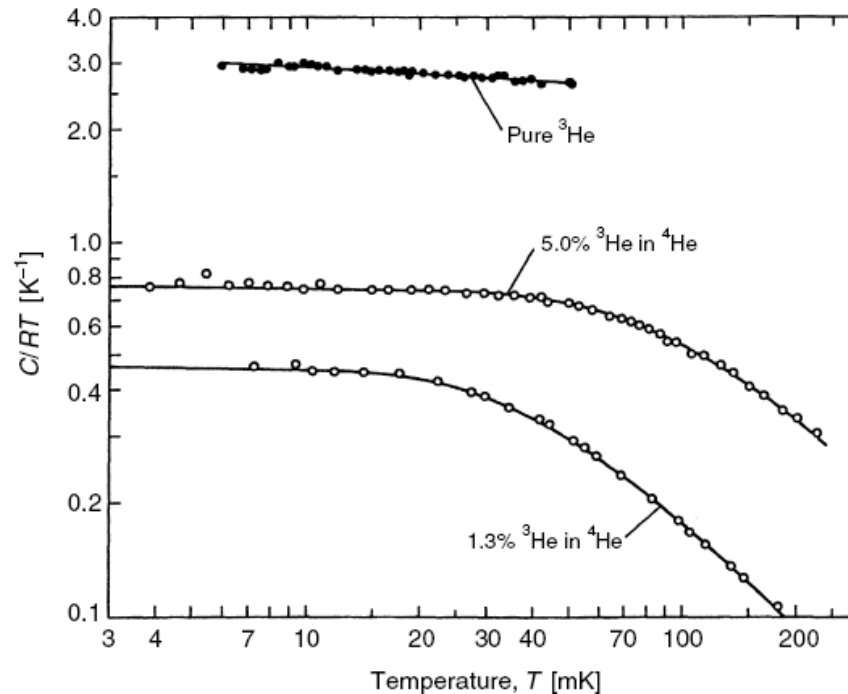
Since $dS = dQ/T$, the entropy is given by integrating:

$$S = \int \frac{dQ}{T} = \int \frac{dU}{T} = \int \frac{C}{T} dT$$

In order to obtain the entropy, we need know the specific heat capacity as a function of temperature.

Heat change of mixing

We start with the concentrated phase, which we shall treat as pure helium-3.



J.C. Wheatley: Am. J. Phys. 36, 181 (1968)
A.C. Anderson, et al: Phys. Rev. Lett. 16, 263 (1966)

The graph above shows the experimental measurements on the specific heat for pure helium-3 on top, and two mixtures below.

Heat change of mixing

By fitting the data, we obtain for pure helium-3 an approximate formula for the specific heat:

$$C_3 = 22T \text{ J}/(\text{mol K})$$

Note the graph only shows measurement up to 40 mK, so this formula is only valid below that.

The entropy for the concentrated phase is therefore

$$S_C = \int_0^T \frac{C_3(T')}{T'} dT' = \int_0^T 22 dT' = 22T \text{ J}/(\text{mol K})$$

In principle, we can use the same method to find the entropy for helium-3 in the dilute phase. Unfortunately, we would need this for different concentrations, and there are not many measurements available.

We have seen earlier that the specific heat formulae for a Fermi gas can be used ...

In the dilute phase, the specific heat for helium-3 may be estimated by the Fermi gas formula

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R$$

Suppose that the concentration of the mixture in a dilution refrigerator is close to 6.6%. Using the effective mass of $m^* = 2.45m_3$, and the formula for the Fermi energy, we would get

$$C_3 = 106T \text{ J}/(\text{mol K})$$

So the entropy for helium-3 in the dilute phase is

$$S_D = \int_0^T \frac{C_3(T')}{T'} dT' = \int_0^T 106 dT' = 106T \text{ J}/(\text{mol K})$$

Heat change of mixing

We have obtained the entropies in the two phases:

$$S_C = 22T \text{ J/(mol K)}$$
$$S_D = 106T \text{ J/(mol K)}$$

We can now find the heat change of mixing:

$$H_D - H_C = T(S_D - S_C) = T(106T - 22T) = 84T^2 \text{ J/mol}$$

Note that this change is positive. This implies an increase in internal energy.

Because of conservation of energy, heat energy has to be absorbed from the surrounding. Hence the cooling.

The cooling power is then

$$\dot{Q} = \Delta H = \dot{n}_3(H_D - H_C) = 84\dot{n}_3T^2 \text{ W}$$

where \dot{n}_3 is the number of moles per second of helium-3 moving from concentrated to dilute phase.

The cooling power is proportional to T^2 , as we have seen early on.

For typical values of $\dot{n}_3 = 100\mu\text{mol s}^{-1}$ and $T = 10 \text{ mK}$, we find

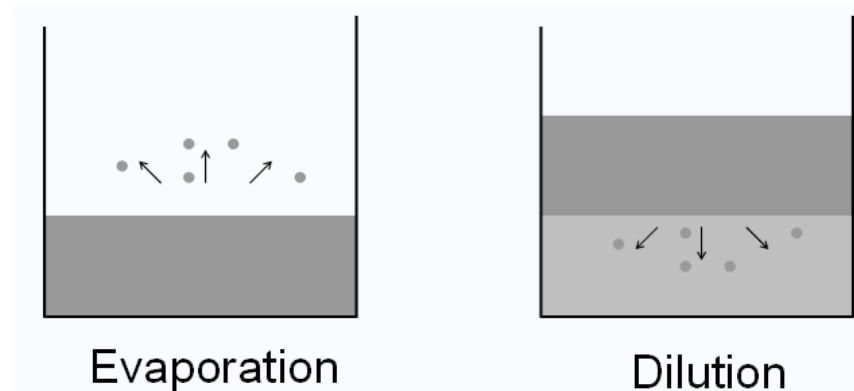
$$\dot{Q} = 1 \mu\text{W}.$$

Realisation of the dilution refrigerator

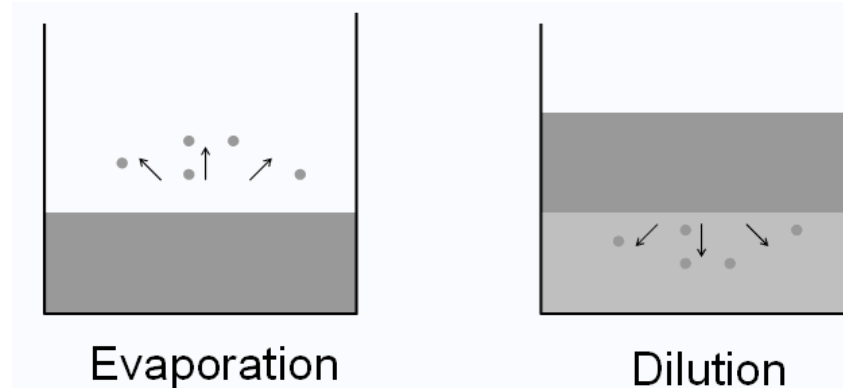
Realisation of the dilution refrigerator

The use of the dilution process for cooling is similar in concept to helium evaporation:

1. In evaporation, cooling takes place when helium atoms move from the liquid to the vapour phase.
2. In dilution, cooling takes place when helium-3 atoms move from the concentrated to the dilute phase.



Realisation of the dilution refrigerator

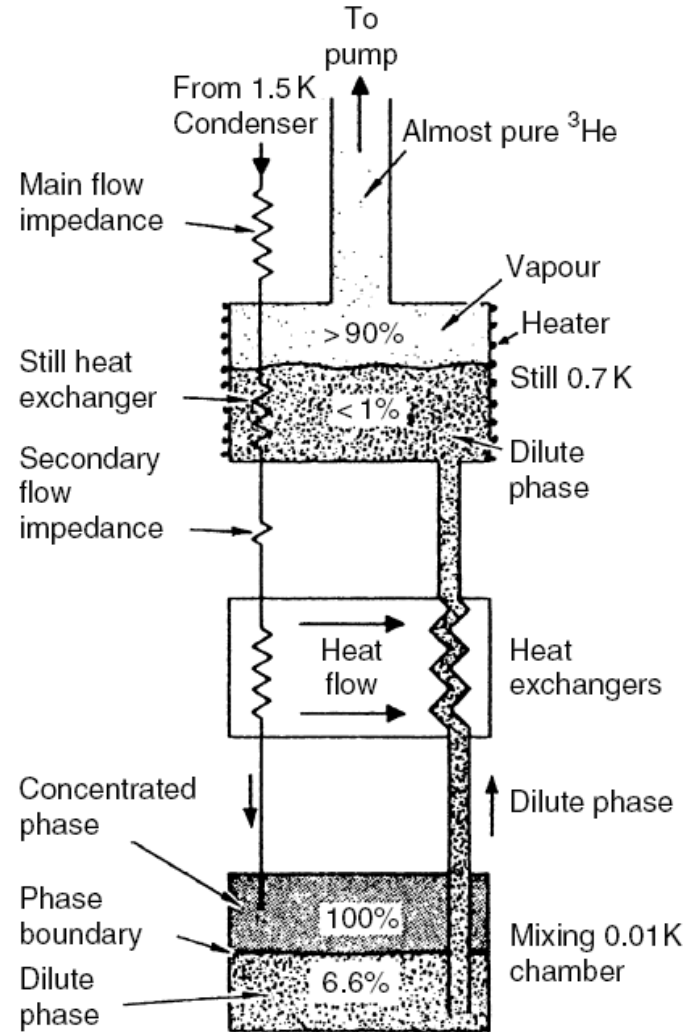


In dilution cooling, there are a number of practical issues:

1. One is that the dilute phase is below the concentrated phase, so helium-3 atoms move down instead of up.
2. Another is the need to find a way to "pump" out the helium-3 from the dilute phase.

Realisation of the dilution refrigerator

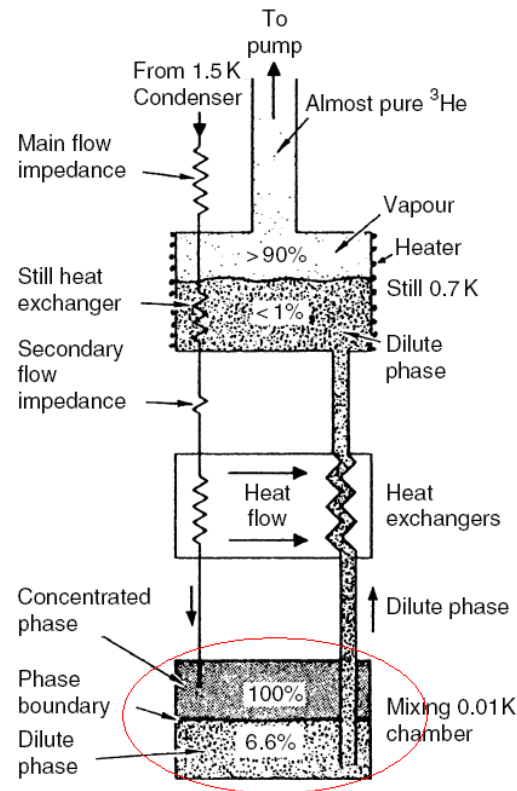
This is a schematic diagram. It would be housed inside a vacuum chamber that is immersed in a helium-4 bath at 4.2 K.



Pobell (2007)

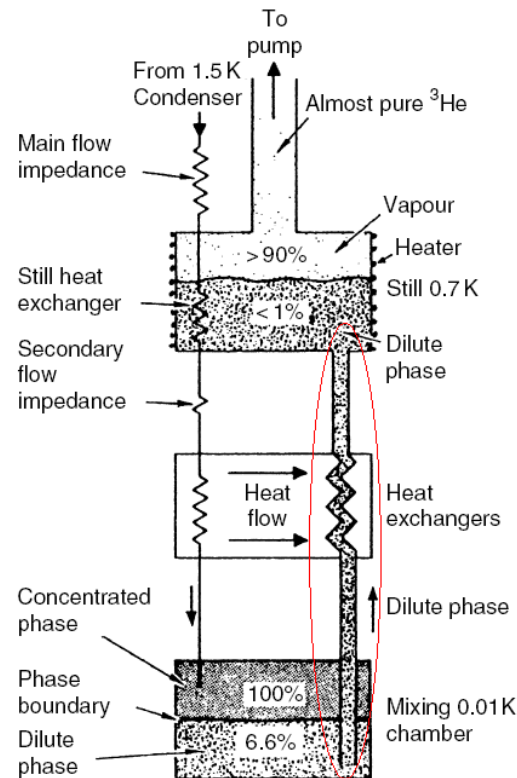
Realisation of the dilution refrigerator

The mixing chamber at the bottom contains the two layers - concentrated phase on top, dilute phase below.



Realisation of the dilution refrigerator

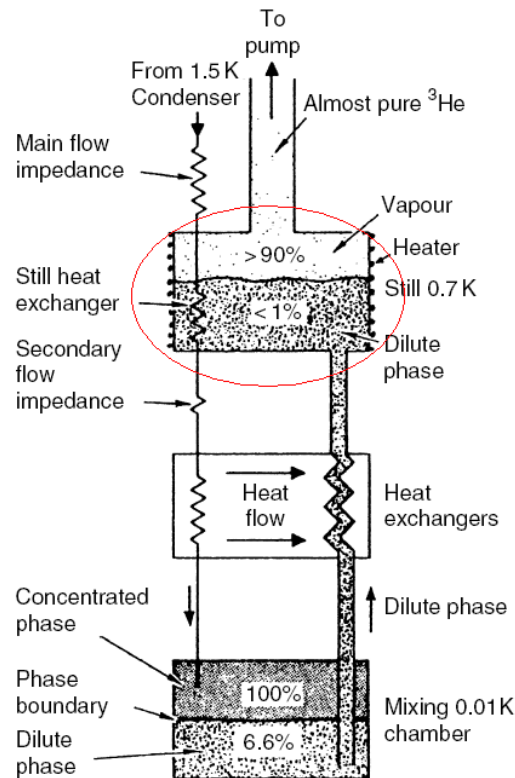
A tube leads from the dilute phase to a chamber at the top, called the "still."



The still is connected to a pump.

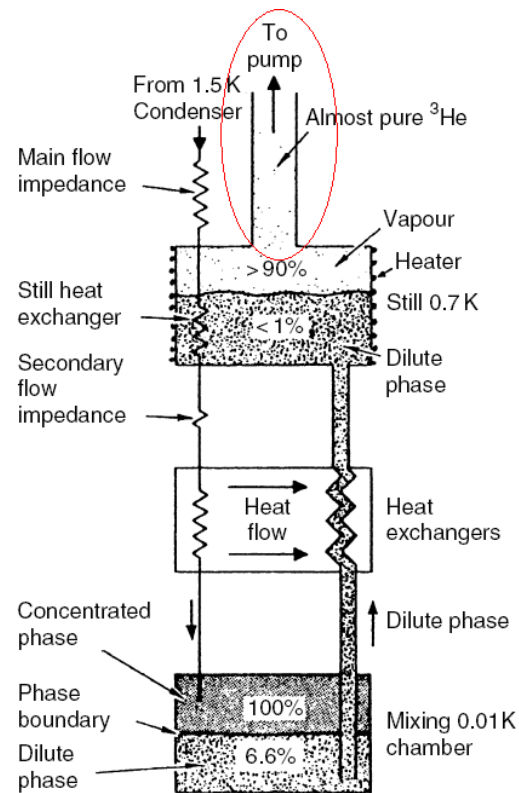
Realisation of the dilution refrigerator

The dilute phase is pushed up from the mixing chamber by pressure, and partly fills the still.



Realisation of the dilution refrigerator

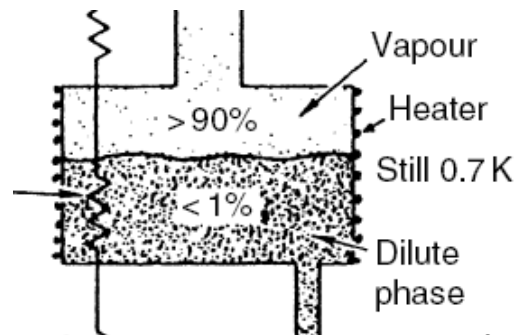
The still is connected to a pump.



Realisation of the dilution refrigerator

In the still, the temperature is maintained at 0.7 K. There, the dilute phase vaporises.

What happens is essentially fractional distillation. The vapour would contain a higher concentration of the lower boiling fraction. In this case, that would be helium-3.

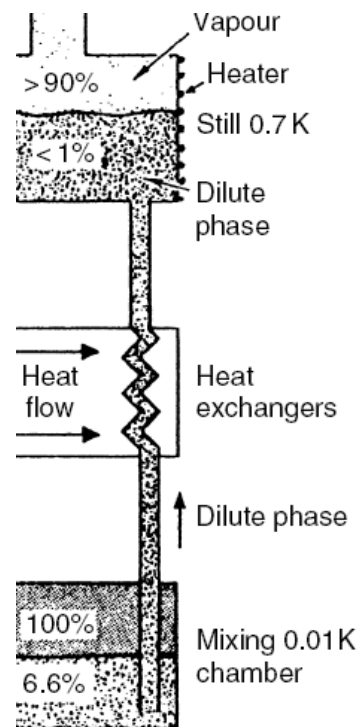


The resulting vapour has more than 90% of helium-3. As a result, the concentration of helium-3 in the mixture falls.

Realisation of the dilution refrigerator

The vapour is high concentrated in helium-3. As the vapour is pumped out, the helium-3 concentration in mixture that is in the still falls.

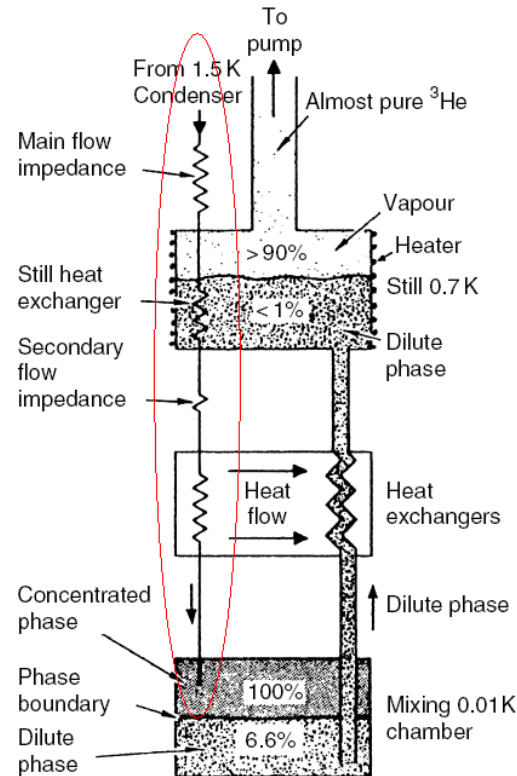
This causes the helium-3 from the mixture in the mixing chamber below to diffuse up to the mixture above.



This is how helium-3 in the mixture that is in the mixing chamber is "pumped" out.

Realisation of the dilution refrigerator

To maintain a continuous flow of helium-3, we must replace the helium-3 in the mixing chamber continuously. In order to do this, the helium-3 that is pumped out from the still is recycled.

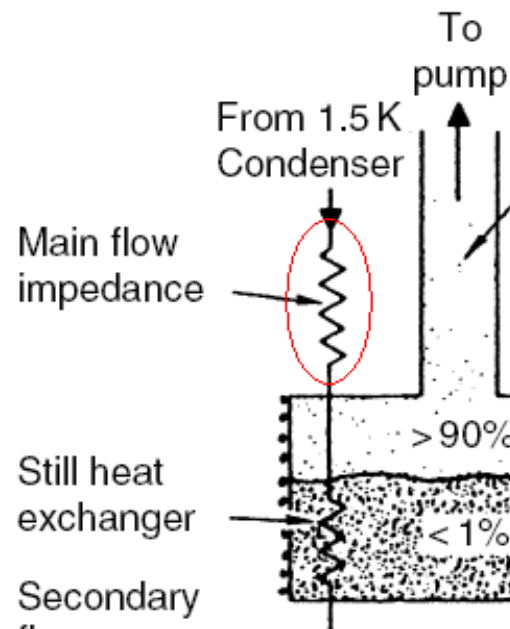


It is passed back in to the mixing chamber through a tube. This tube is represented by the vertical black line on the left side of the diagram that runs through all the chambers.

Realisation of the dilution refrigerator

But first, the recycled helium-3 gas must be precooled to 1.5 K. This is done in a helium-4 bath that is not shown.

In order to liquify the helium-3 gas, the pressure must be increased. This is achieved by making a section of the tube very narrow.

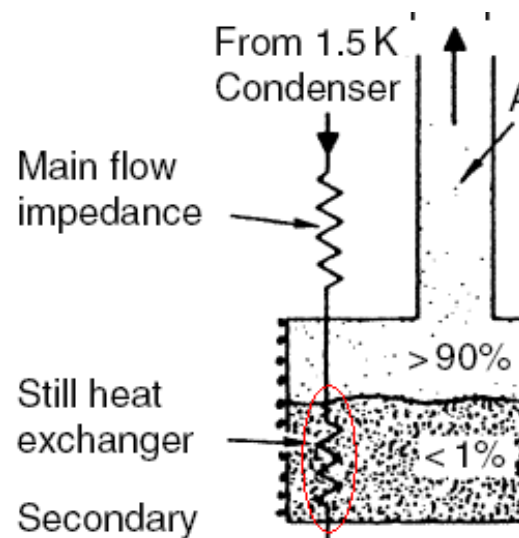


This narrow section is labeled the Main Flow Impedance in the diagram.

Realisation of the dilution refrigerator

The flow impedance increases the pressure of the helium-3 gas. As a result, it becomes a liquid before it enters the still.

In the still, the tube is designed to be in close thermal contact with the mixture. This helps to cool down the incoming helium-3 liquid.

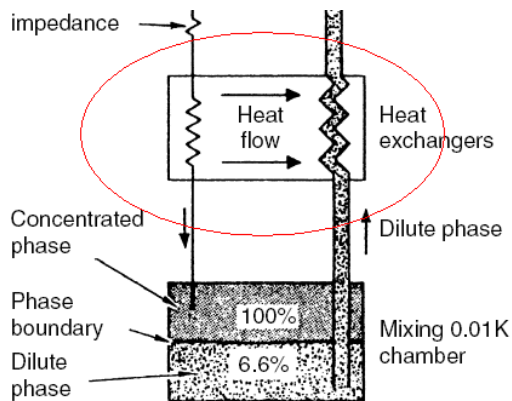


This part of the tube is labeled Still Heat Exchanger in the diagram.

Realisation of the dilution refrigerator

After leaving the still, the tube carries the incoming liquid helium-3 down to the mixing chamber.

On its way down, the tube is designed to be in close contact with the cold mixture leaving the mixing chamber. This helps to cool the incoming helium-3 further.

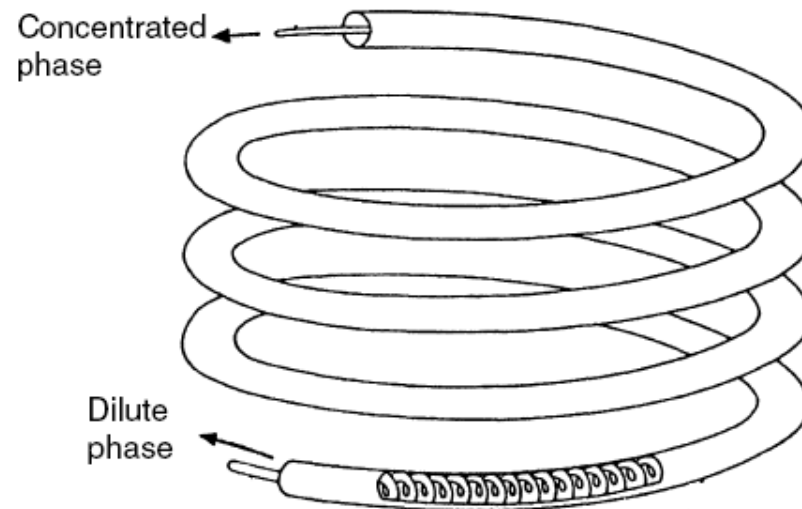


This part of the tube is labeled Heat Exchangers in the diagram.

Realisation of the dilution refrigerator

It should be mentioned that heat exchanger is a very important part of any low temperature refrigerator.

By fully utilising the "cold" coming out from the mixing chamber, the heat exchange saves a lot of time and a lot of liquid helium.

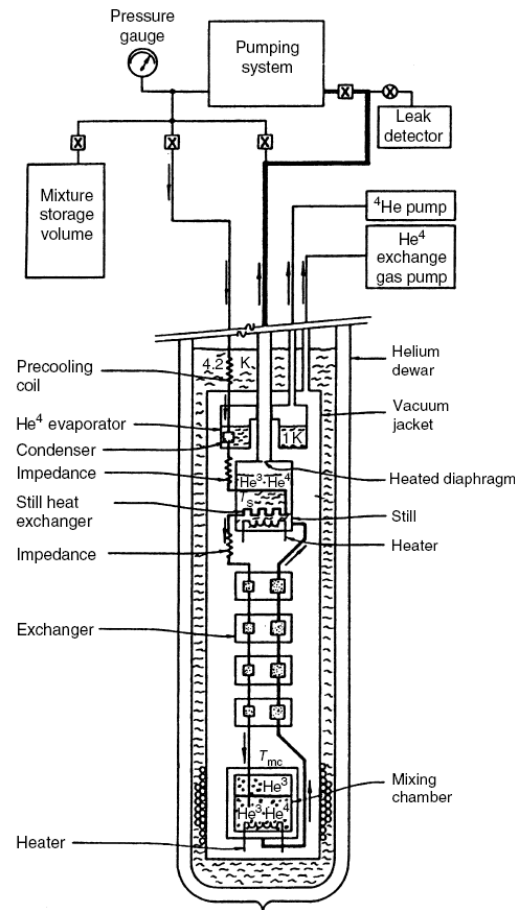


Y. Oda, G. Fujii, T. Ono, H. Nagano: Cryogenics 23, 139 (1983)

In this way, it allows the refrigerator to cool down faster, and to reach a lower temperature.

Realisation of the dilution refrigerator

This is a more complete schematic diagram. It shows the surrounding vacuum chamber, helium-4 bath, pumps and other things that are needed.

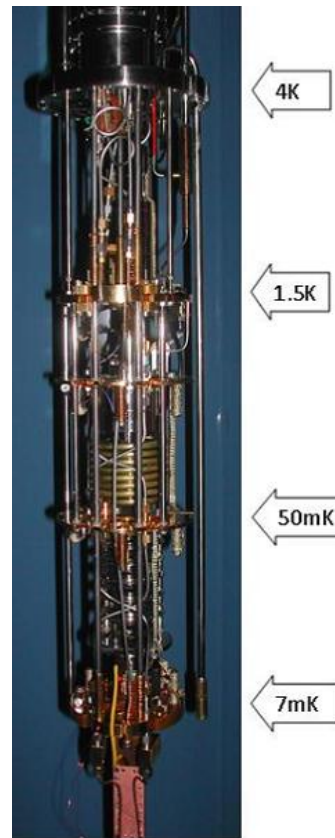


J.C. Wheatley, O.E. Vilches, W.R. Abel: Physics 4, 1 (1968)

Examples of dilution refrigerators

Examples of dilution refrigerators

This is a dilution refrigerator used in the semiconductor physics group in the Cambridge physics department.

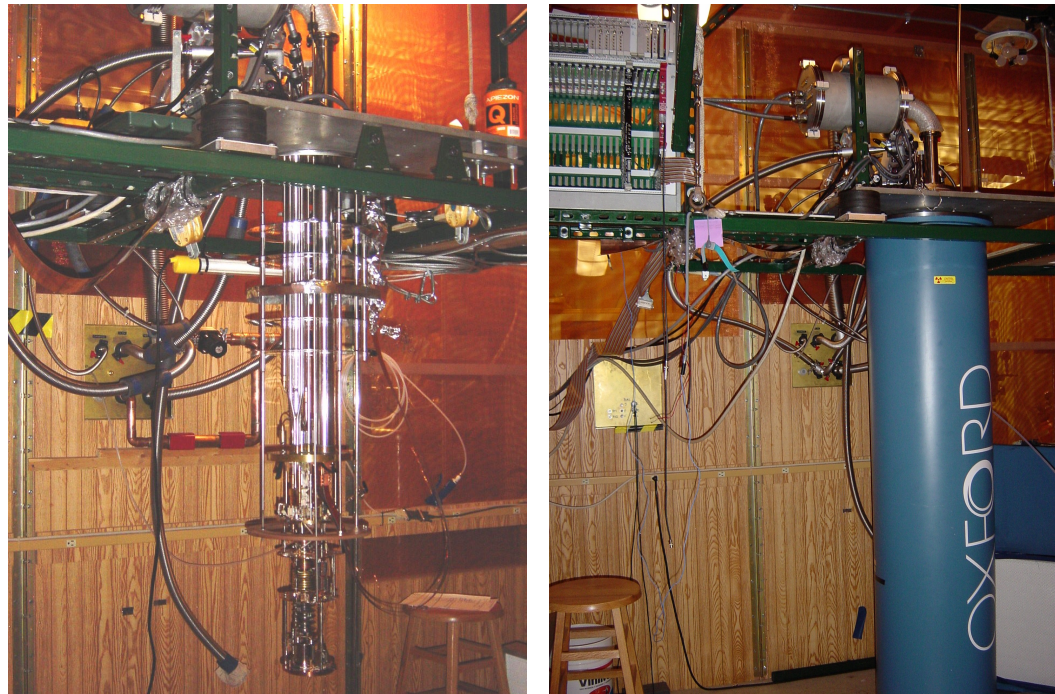


<http://www.phy.cam.ac.uk/research/sp/cryo.php>

You should be able to tell from the temperatures indicated what the various parts are.

Examples of dilution refrigerators

This is a dilution refrigerator used to test detectors for dark matter coming to earth from outer space.



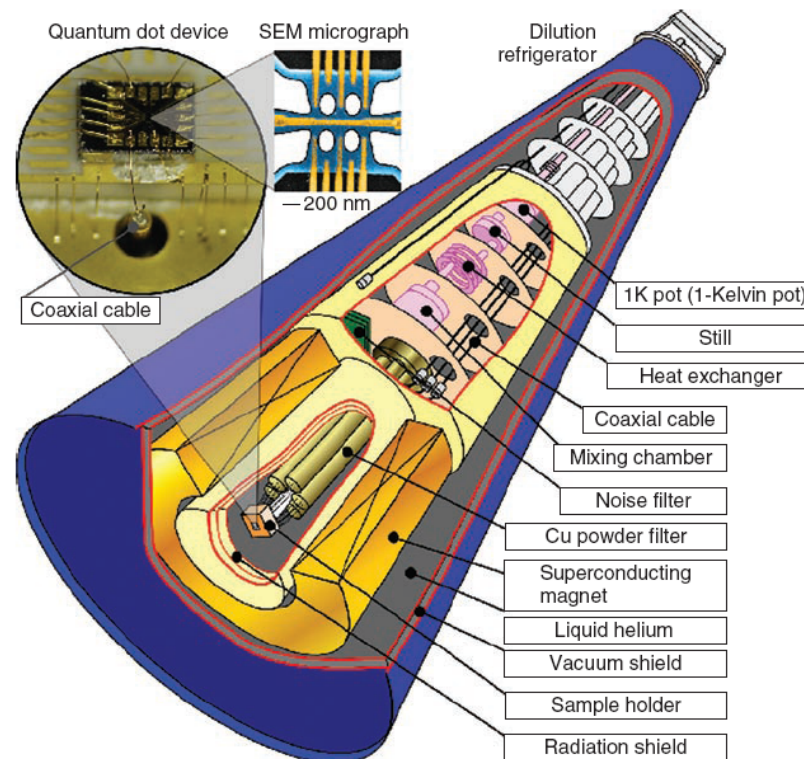
Case Western Reserve University (USA)
<http://cdms.case.edu/learn/caselearn/fridge.html>

On the left is the dilution refrigerator.

On the right, the refrigerator is inserted into a helium cryostat.

Examples of dilution refrigerators

This is a dilution refrigerator used for experiments in quantum computing.



T. Fujisawa, NTT Technical Review, Vol. 6 No. 1 Jan. 2008
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Notice how the sample is attached to the lower end of the refrigerator.

What we have learnt so far.

1. The ^3He - ^4He mixture undergoes phase separation when cooled below 0.87 K, giving 2 phases.
2. The specific heat of a helium-3 atoms is higher in the dilute phase than in the concentrated phase. If the atom goes from the concentrated to the dilute phase, it results in the "production of cold,"
3. There is a non-zero solubility of ^3He in ^4He even at 0 K. This leads to a cooling power which decreases with T^2 . This is much higher than the cooling power of evaporation, which falls exponentially.
4. In the vapour above a ^3He - ^4He mixture, the vapour pressure of helium-3 is much higher than that of helium-4. This makes it possible to circulate nearly pure helium-3.

Worked Examples

Example 1

Calculate the Fermi temperature of liquid helium-3.

[You are given the following:

The volume V_m of 1 mole of liquid helium-3 is 36.84 cm^3 .

The effective mass is $m^* = 2.78m_3$,

where m_3 is the mass of the helium-3 atom.

The atomic mass unit u is $1.6605 \times 10^{-27} \text{ kg}$.]

The Fermi temperature is given by

$$T_F = \frac{E_F}{k_B},$$

where the Fermi energy is

$$E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}.$$

Using the molar volume V_m that is given, N would be the Avogadro's constant N_A .

The mass of the helium-3 atom is $3u$, and the mass m in the formula for the Fermi energy would be the effective mass

$$m^* = 2.78m_3 = 2.78 \times 3u,$$

where u is the atomic mass unit.

Substituting all these values into the Fermi energy equation, we find

$$E_F = 5.31 \times 10^{-23} \text{ J.}$$

Then the Fermi temperature is

$$T_F = \frac{E_F}{k_B} = 1.79 \text{ K.}$$

Example 2

At which temperature would the Fermi heat capacity of liquid helium-3 reach the classical ideal gas value

$$C_P = \frac{5}{2}R$$

if its heat capacity continued to vary with temperature, as expressed by the Fermi gas formula

$$C_3 = \frac{\pi^2}{2} \frac{T}{T_F} R.$$

[You are given the following:

The Fermi temperature T_F of liquid helium 3 is 1.79 K.

The ideal gas constant $R = 8.315 \text{ J}/(\text{mol K}).]$

Substituting the given values of T_F and R into the Fermi gas formula

$$C_3 = \frac{\pi^2 T}{2 T_F} R,$$

we get

$$C_3 = 22.9T \text{ [J/(mol.K)]}.$$

This would become equal to the classical value

$$C_P = \frac{5}{2} R$$

when

$$22.9T = \frac{5}{2} R.$$

Solving for T gives 0.91 K.

Remark: Notice that this is smaller than the Fermi temperature of 1.79 K. So for the helium-3 to behave like a Fermi gas, the temperature must be much smaller than this value.

Example 3

Assuming that liquid helium-3 is a Fermi gas, show that the entropy of liquid helium-3 is given by

$$S = C_3$$

where C_3 is the heat capacity. Show that the enthalpy change is given by

$$\Delta H = T(C_D - C_C)$$

where C_D and C_C are the heat capacities in the dilute phase and the concentrated phase respectively.

The Fermi gas heat capacity is given by

$$C_3 = \frac{\pi^2 T}{2 T_F} R.$$

The entropy is

$$S = \int \frac{dQ}{T} = \int \frac{C_3 dT}{T}.$$

Substituting, we get

$$S = \int \frac{\pi^2}{2 T_F} R dT = \frac{\pi^2 T}{2 T_F} R$$

which is the same as C_3 .

At phase equilibrium, the condition is

$$H_C - TS_C = H_D - TS_D$$

so the enthalpy change is

$$\Delta H = H_D - H_C = T(S_D - S_C).$$

Since the entropy is equal to the heat capacity, we have

$$\Delta H = T(C_D - C_C).$$

Example 4

The heat capacity of a Fermi gas is given by

$$C = \frac{\pi^2}{2} \frac{T}{T_F} R.$$

Assuming that a mixture of helium-3 in liquid helium-4 behaves as a Fermi gas, show that

$$C \propto \frac{r}{x^{2/3}}$$

where r is the effective mass ratio, and x is the fractional concentration in the dilute phase.

The heat capacity is inversely proportional to the Fermi temperature, as we can see from its formula:

$$C = \frac{\pi^2 T}{2 T_F} R.$$

The Fermi temperature T_F is directly related to the Fermi energy E_F , which is

$$E_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}.$$

This is in turn inversely proportional to the mass m . It is also proportional to $(N/V)^{2/3}$, where N/V is the molar concentration.

We are given that r is the effective mass ratio, and x is the fractional concentration. So

$$E_F \propto \frac{x^{2/3}}{r}$$

We have found that

$$E_F \propto \frac{x^{2/3}}{r}$$

Therefore, since the heat capacity C is inversely proportional to the Fermi temperature T_F , we have

$$C \propto \frac{r}{x^{2/3}}.$$

Remark: This tells us the heat capacity of helium-3 is higher when the concentration is lower, and that it is proportional to the effective mass.

Example 5

The effective mass ratio of pure helium-3 increases from about 2.8 at saturated vapour pressure to about 4.6 at 20 bar. In the same pressure range, the effective mass ratio for a saturated mixture increases from about 2.4 to 2.8. How would the numerical value for the cooling power of a dilution refrigerator given by

$$\dot{Q} = 84n_3T^2$$

change if this refrigerator were operated at 20 bar.

[You are given that the heat capacity of helium-3 is related to the effective mass ratio r , and the fractional concentration x , by

$$C \propto \frac{r}{x^{2/3}}.$$

]

The cooling power is proportional to both the enthalpy change ΔH and the flow rate \dot{n}_3 :

$$\dot{Q} \propto x \Delta H.$$

The enthalpy change in turn depend on the entropies of the concentrated (C) and the dilute (D) phases:

$$\Delta H = T(S_D - S_C).$$

From the previous exercise, we know that the entropy is equal to the heat capacity for the Fermi gas, so that

$$\Delta H = T(C_D - C_C).$$

We are given that

$$C \propto \frac{r}{x^{2/3}}.$$

Let the constant of proportion be C_1 . Then the enthalpy may be written as

$$\Delta H = TC_1 \left(\frac{r_D}{x_D^{2/3}} - \frac{r_C}{x_C^{2/3}} \right).$$

In this equation, we have continued to use the subscript D to denote dilute phase and C for concentrated phase.

We are given that at saturated vapour pressure, the effective mass ratios are

$$r_C = 2.8, \text{ and } r_D = 2.4.$$

In the concentrated phase, the concentration is nearly 100%. In the dilute phase has a limiting concentration of 6.6%. So

$$x_C = 1, \text{ and } x_D = 0.066.$$

Substituting these into the above equation for ΔH , we find

$$\Delta H = 11.9TC_1.$$

We are given that at the pressure of 20 bar, the effective mass ratios change to

$$r_C = 4.6, \text{ and } r_D = 2.8.$$

Substituting these into the above equation for ΔH , we find

$$\Delta H = 12.5TC_1.$$

So the enthalpy change ΔH increases by about 5%.

Since the cooling power is proportional to the enthalpy change ΔH :

$$\dot{Q} \propto x\Delta H,$$

it would also increase by 5%.

Remark:

1. This shows that the cooling power may be increased by increasing the pressure.
2. We have assumed that the limiting concentration in the dilute phase remains the same. In fact it would increase. This increases the flow rate and gives a greater increase in the cooling power.