Statistical and Low Temperature Physics (PHYS393)

8. Superconductor

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Learning Aims: You will learn to

Describe the behaviours a superconductor: zero resistance, Meissner's effect, and heat capacity. Explain the significance of the heat capacity behaviour.

Discuss why Faraday's law of electromagnetic induction cannot explain Meissner's effect. Explain Meissner's effect using a macroscopic wavefunction.

Explain the cause of London penetration depth. Derive a formula for it.

Explain the significance of the isotope effect. Suggest how electrons can attract each other.

Describe the effect of electron movement on positive ions in metal. Explain how the motion of the ions are related to Debye frequency.

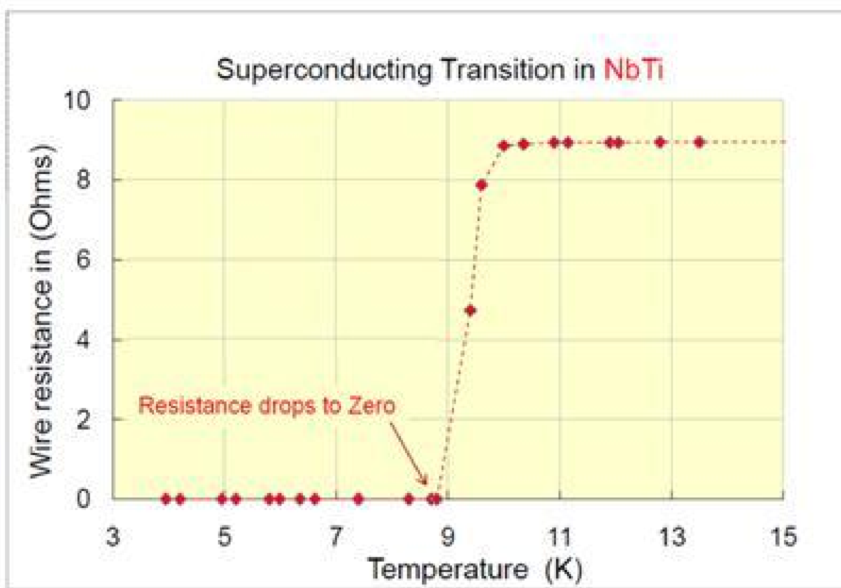
Describe how electron movement leaves behind a trail of displaced ions and explain how to calculate the length of this trail. Suggest how this relates to the size of the Cooper pair.

Explain how the idea of superfluid is applied to the Cooper pair.

Resistance, magnetic field and heat capacity observations

Metals conduct electricity. Normally, there is always some resistance, however small. In some materials, this resistance suddenly falls to zero below a certain temperature. In 1911, Kamerlingh Onnes discovered that this happened with mercury below 4.2 K

Here is an example of superconducting transition in Niobium-Titanium wire, observed in 2010 by two high school students at Daresbury Laboratory.

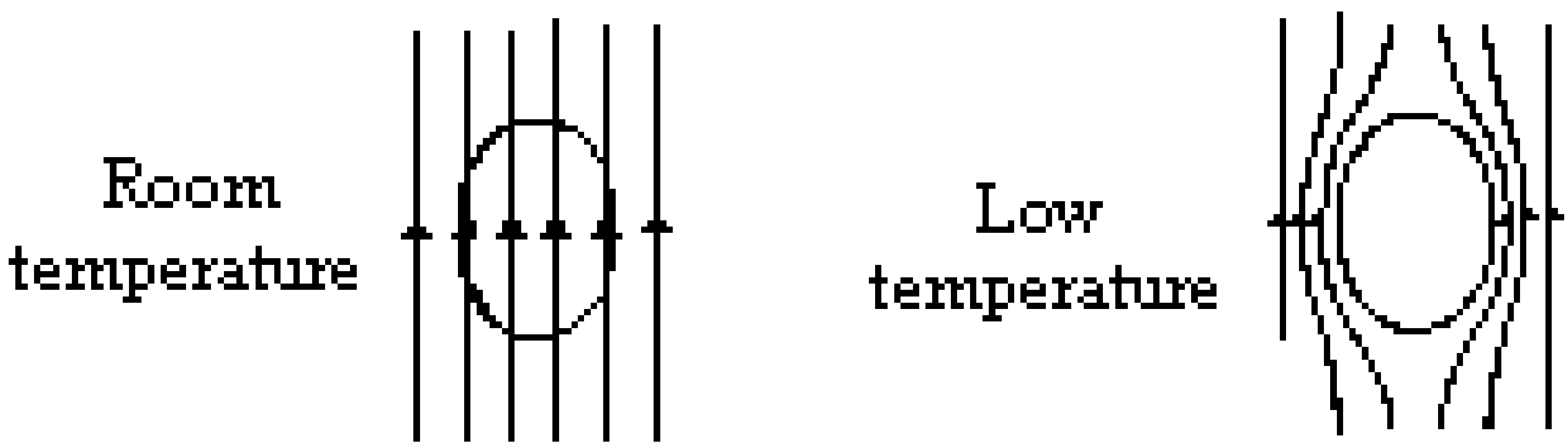


These are examples of metals that become superconducting at very low temperatures. The transition temperatures are close to liquid helium temperature.

Material	T-Critical
Gallium	1.1 K
Aluminum	1.2 K
Indium	3.4 K
Tin	3.7 K
Mercury	4.2 K
Lead	7.2 K
Niobium	9.3 K
Niobium-Tin	17.9 K

Another class of superconductors, called high temperature superconductors, are ceramic. These can become superconducting at temperatures higher than liquid nitrogen temperature.

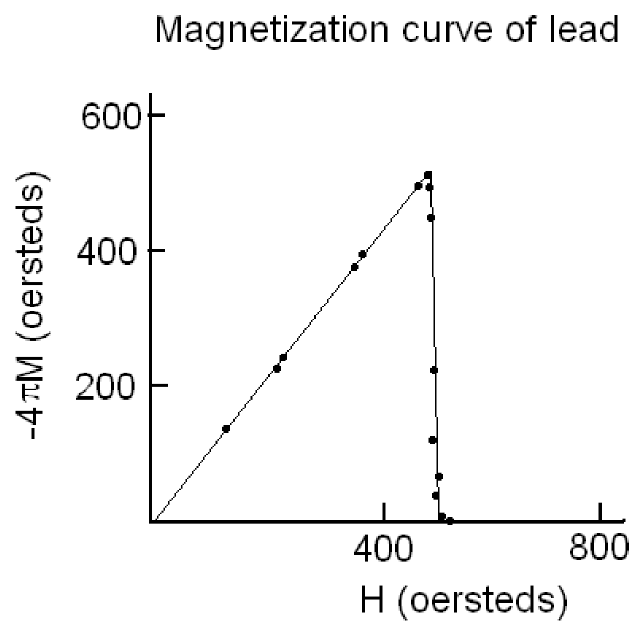
When the resistance drops to zero, the superconductor all expels all magnetic field from its body.



http://www.materia.coppe.ufrj.br/sarra/artigos/artigo10114/index.html

The field inside the body of a superconductor can be obtained by inserting it in a coil and measuring the induced voltage.

This graph shows the magnetisation of lead in liquid helium, plotted against the applied field.



Livingston, Physical Review, vol. 129 (1963), p. 1943

Below a certain critical field, the magnetisation is equal and opposite to the applied field. So the applied field in the conductor is cancelled. Measurement of magnetisation is how we know that the resultant field inside the superconductor is zero.

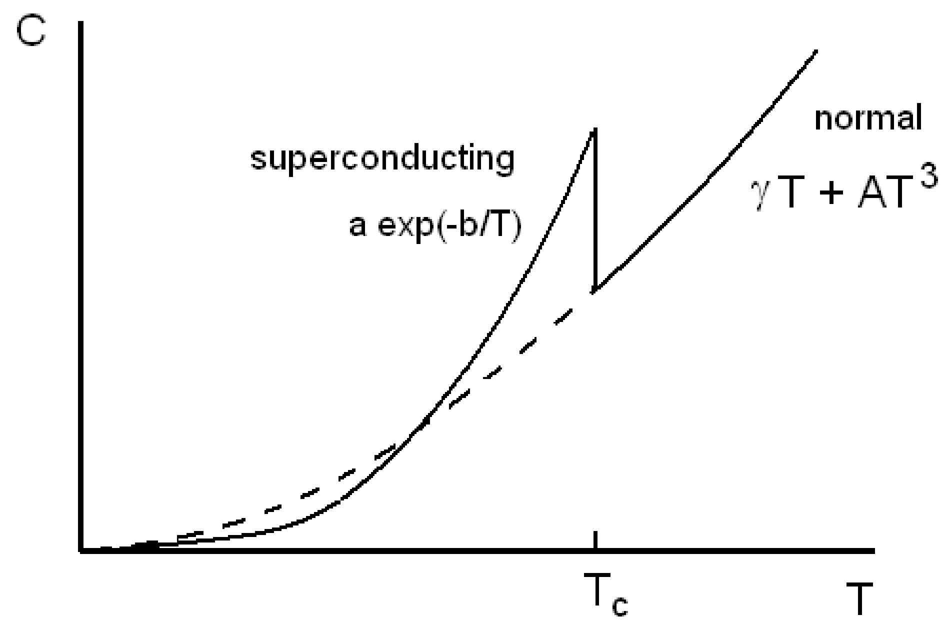
The expulsion of magnetic field from a superconductor is called is Meissner effect. A popular demonstration is the levitation of a superconductor above a magnet.

Note that this results from the opposing magnetisation. You can get levitation by putting a north pole of a magnet on top of the north pole of another magnet. So levitation does not just happen to superconductors only.

Recall the heat capacity of a normal metal:

$$C_v = \gamma T + AT^3.$$

Measurements show that for superconductors, this changes completely below the transition temperature. This graph is the result of measurement for the niobium metal.



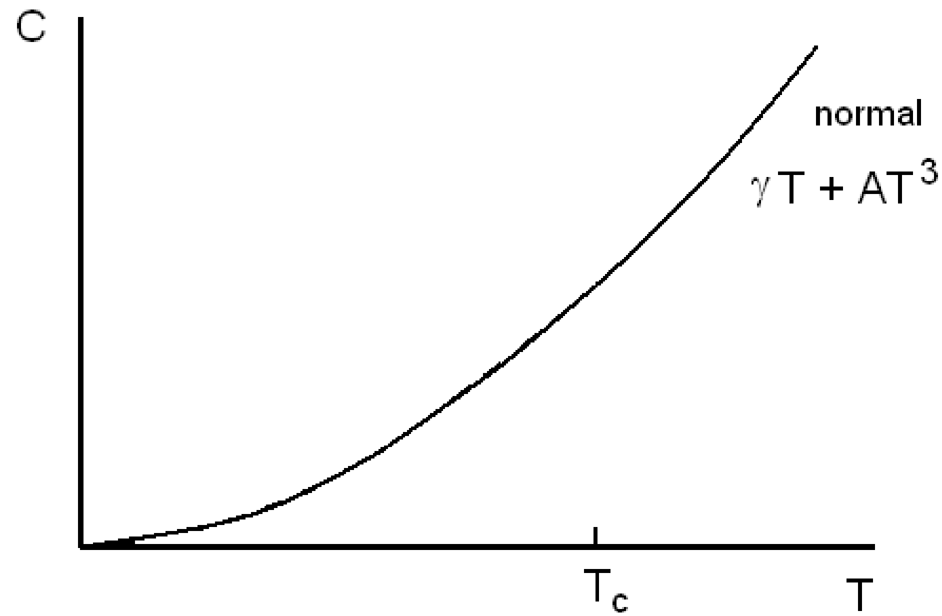
Brown, et al, Physical Review, vol. 92 (1953), p. 52

Niobium become superconducting below 9.5 K. It is possible to prevent it from becoming superconducting by applying a sufficiently large enough magnetic field.

We know from the Meissner effect that a niobium expels all magnetic field. However, if the field is strong enough, it can "force" its way into the superconductor. This destroys the superconductivity and returns the niobium to a normal conducting state - even if temperature is below 9.5 K.

Using this property, it is possible to select between the normal and the superconducting state.

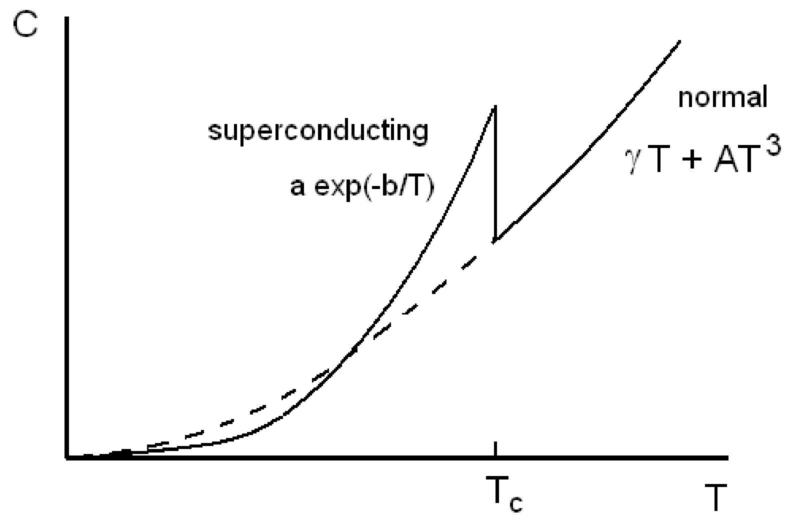
If we select the normal conducting state of niobium by applying a strong magnetic field, we would measure the curve labelled "normal."



This follows the "normal" behaviour of

$$C_v = \gamma T + AT^3.$$

If we do not apply any magnetic field, we get the superconducting state. Then we would get the curve labelled "superconducting".



If we subtract the phonon contribution of AT^3, we would find that the curve is closer to the exponential form:

$$C = a\,\mathrm{exp}(-b/T)$$

for some constants a and b. This looks like the Boltzmann distribution.

Explanation using macroscopic wavefunction

In 1937, Fritz London suggested that if the electrons in a superconductor somehow forms a macroscopic wavefunction.

Using this assumption, London was able to explain it expels all magnetic field. To understand this, we first need to appreciate why expulsion of the magnetic field is strange.

Suppose the resistance going to zero is the only change in a metal. Consider what happens if we now bring a magnetic to the metal.

The change in magnetic flux through the metal induces an electric current, according to Faraday's law.

According to Lenz's law, the current would flow in such a way as to produce a magnetic field of its own that opposes the incoming field.

In a metal with resistance, this induced current would quickly slow down to zero. The induced field becomes zero, and only the incoming field remains in the body of the metal.

If the metal has no resistance, the induced current continues to flow. The induced flux has to be opposite to the incoming flux. Therefore they cancel, and the field in the body becomes zero. In this way, the field is "expelled."

It looks like we have just “explained” the Meissner effect. However, let us now look at what happens if the magnet is already there before cooling.

We start with a normal metal with a magnetic field going through the body. Then we cool this down and the resistance falls to zero.

According to Faraday’s law, since there is no change in magnetic flux, no current is induced. So the original field from the magnet remains in the body.

In a real superconductor, we know from the Meissner effect that, even in this case, the magnetic field must be expelled.

This shows that there is something different about a superconductor that the familiar laws of electromagnetism cannot explain.

We shall now see how a macroscopic wavefunction, ψ, can explain the Meissner effect.

Recall the operator in quantum mechanics for momentum:

$$-i\hbar\frac{d\psi}{dx} = p_x\psi$$

where p is the momentum mv.

In the presence of an electromagnetic field, this is changed to

$$-i\hbar\frac{d\psi}{dx} = (mv + qA)\psi$$

where A is the vector potential and q the charge of the particle.

Both equations are quantum mechanical postulates that have been shown to give correct results in physics.

In order to use the vector potential, lets review its meaning. It is defined by

$$\nabla \times \mathbf{A} = \mathbf{B},$$

where $\mathbf{B}$ is the magnetic field.

(The corresponding relation between the electric field and electric potential is $-\nabla\phi = \mathbf{E}$.)

Applying Stoke's theorem and integrating around any loop C:

$$\int_C \mathbf{A}.d\mathbf{l} = \int_S \mathbf{B}.d\mathbf{S},$$

where S is any surface enclosed by the loop.

The right hand side is the magnetic flux, Φ.

Let us now return to the quantum mechanical equation:

$$-i\hbar\frac{d\psi}{dx} = (mv + qA)\psi.$$

Recall the wavefunction we used for superfluids:

$$\psi = e^{-i\phi(x)}$$

where $\phi(x)$ is the phase. Substituting into the equation, we get

$$\hbar\frac{d\phi}{dx} = mv + qA.$$

This relation along a straight line in x can be extended in a simple way to any path or loop in 3D.

Consider a loop in a superconductor of length L enclosing an area S. Integrating along this loop, we get

$$\hbar\Delta\phi = m\int_L \mathbf{v}.d\mathbf{l} + q\int_L \mathbf{A}.d\mathbf{l}.$$

$$\hbar\Delta\phi = m\int_L \mathbf{v}.d\mathbf{l} + q\int_L \mathbf{A}.d\mathbf{l}.$$

The phase change $\Delta\phi$ is zero or a multiple of 2π, because the wavefunction returns to the same value after one loop.

The integral over $\mathbf{A}$ gives the magnetic flux Φ.

The velocity $\mathbf{v}$ is related to the current density $\mathbf{J}$ by

$$\mathbf{J} = \rho q\mathbf{v},$$

where ρ is the number density of the electrons. The above equation then becomes

$$\hbar\Delta\phi = \frac{m}{\rho q}\int_L \mathbf{J}.d\mathbf{l} + q\Phi.$$

Let us now see how this equation

$$\hbar\Delta\phi = \frac{m}{\rho q}\int_L \mathbf{J}.d\mathbf{l} + q\Phi.$$

can help us understand Meissner's effect.

For a simple lump of metal, the wavefunction would be continuous through the whole volume, so the phase change would be zero. The equation then simplifies to

$$\frac{m}{\rho q}\int_L \mathbf{J}.d\mathbf{l} = -q\Phi.$$

This means that:

> if there is a magnetic field in the macroscopic wavefunction, then is a there is an electric current.

To see why this is special, consider Faraday's law again.

According to Faraday's law, a *change* in magnetic flux is required before a current can be induced.

For a macroscopic wavefunction, the very presence of the flux produces the current. No change in flux is needed!

Let us look at the case of transition to the superconducting state again. Previously, we have not been able to explain the expulsion of the field using Faraday's law.

We can now explain this assuming that a macroscopic wavefunction appears when the metal becomes superconducting, If there is a magnetic field in the metal, it would produce a current. This current would in turn produce a flux.

A more detailed reasoning would show that this wavefunction flux is in the opposite direction to the incoming flux.

London penetration depth

Flux from the wavefunction, or superconducting, current would cancel some of the incoming flux.

The amount cancelled depends on the density of the electrons in the wavefunction. The higher the density, the larger the superconducting current, and more of the incoming flux would be cancelled.

For a uniform external field, this superconducting current would typically be circulating the metal. So it produces the greatest field at the centre, where more cancellation takes place.

For larger electron density, the region of cancellation is also larger. In a typical superconductor, there is sufficent density to expel the incoming field from most of the volume.

In practice, some field would penetrate to a depth of about 100 nm on the surface.

The reason for the penetration depth is that a current is needed to keep the field expelled.

Recall that a field must be present in the macroscopic wavefunction in order to produce the current. As the field gets expelled from the center of the superconductor, the current at the center would also stop.

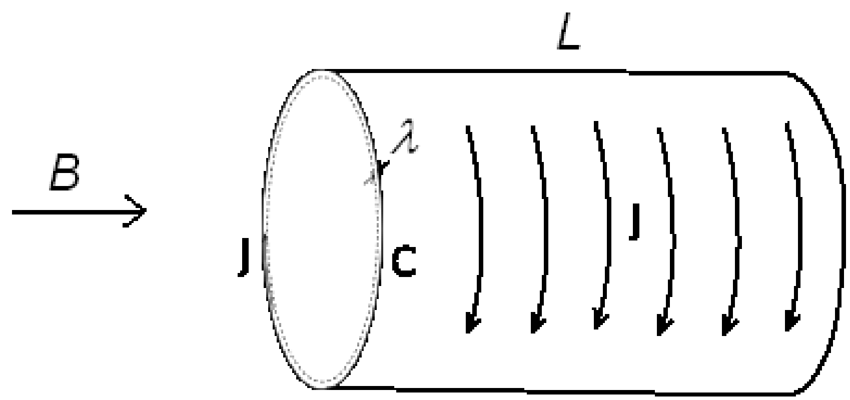
If the field is completely expelled from the metal, there would be no current at all in the metal. Then there would be no opposing flux to cancel the incoming flux. The external flux would come in again and start producing current.

For this reason, a balance would to be reached. The field would penetrate until a depth when there is sufficient current to keep the rest of the volume field free.

Assuming that electrons form a macroscopic wavefunction, Fritz London showed that the magnetic flux Φ in a superconductor is related to the current density J by:

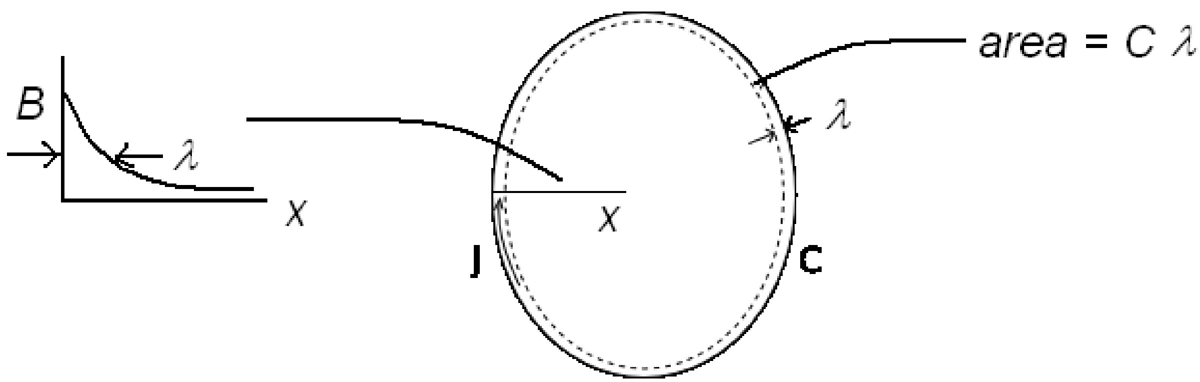
$$\frac{m}{q\rho}\int \mathbf{J}.d\mathbf{l} = q\Phi$$

where m is the mass of the electron, q the charge, and ρ the number density of the electrons. The integral is taken over any closed path, and Φ is the flux enclosed.



Consider a long cylinder with magnetic flux parallel to its axis. Suppose that the current present in a layer at the surface is just enough to cancel the external flux inside.

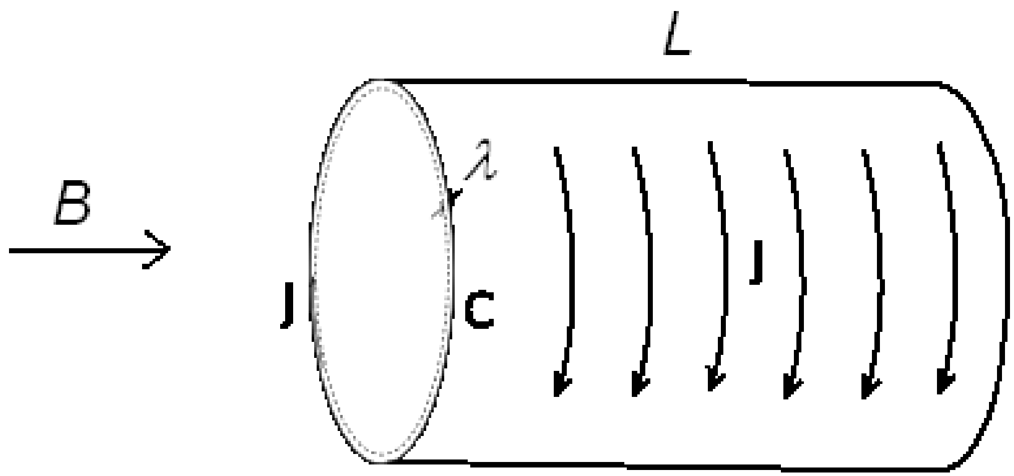
Compared to the surface, the centre of the cylinder is enclosed by more circulating current, which produces the opposing field. So more of the external field would be cancelled, giving a smaller resultant field at the centre.



A graph of the field B versus distance from the surface would look like an exponentially falling curve. The average width of the curve, λ is called the London penetration depth.

Integrating the current along a circumference C, and assuming a uniform current J in the layer, we find

$$\frac{m}{q\rho} JC = qB(C\lambda).$$

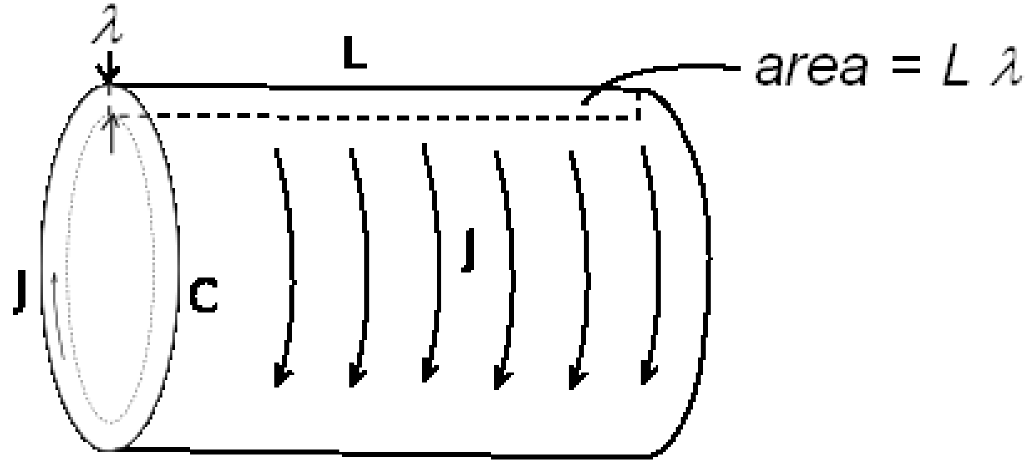


J is unknown. In order to find the thickness λ, notice that the current flows like a solenoid, which has the formula

$$B = \frac{\mu_0 N I}{L}.$$

NI corresponds to the total current. The cross-sectional area of this current in the layer is $L\lambda$. So the current density is

$$J = \frac{\text{current}}{\text{area}} = \frac{NI}{L\lambda}.$$



Combining with the solenoid formula, we get

$$B = \frac{\mu_0 NI}{L} \times \frac{\lambda}{\lambda} = \mu_0 J \lambda.$$

Since the field inside the superconductor is zero, this field produced by the current must be equal and opposite the the external field. Substituting into the previous expression:

$$\frac{m}{q\rho} JC = qB(C\lambda).$$

and rearranging, we find

$$\lambda^2 = \frac{m}{\mu_0 q^2 \rho}.$$

λ is called the London penetration depth. It can be measured by the change in reflection it causes to microwaves falling on the surface. E.g. measurements on Niobium gives an estimate of 340 Å.

Isotope Effect and Cooper Pairs

If electrons repel each other, how can they form a pair?

The clue: In 1950, the superconducting temperature of Mercury was found to be different for different isotopes of Mercury.

Transition temperatures.

Sample	Average mass number	T_0(°K)
1	203.4	4.126
2	202.0	4.143
3	200.7	4.150
4	199.7	4.161

Reynolds, et al, Physical Review, vol. 78 (1950) p. 487

The only difference between isotopes is the number of neutrons in the nuclei. This should not affect the conduction electrons!

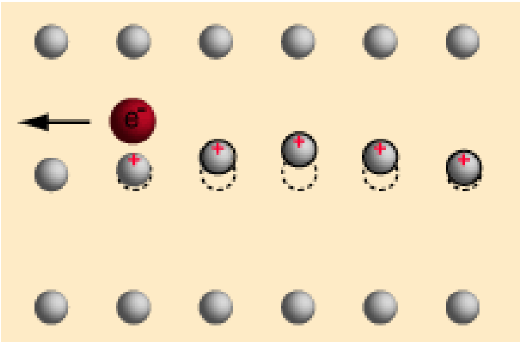
Why do the neutrons change the superconducting temperature?

One possible reason is that the movement of the atoms are somehow involved in causing the superconductivity.

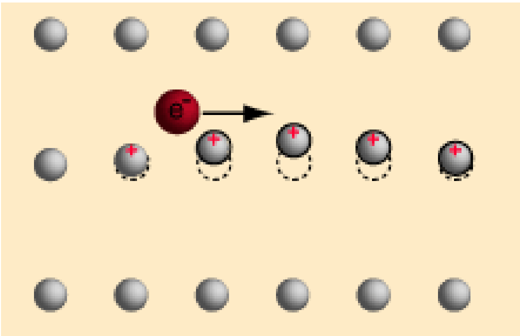
More neutrons means more mass. This would result in slower movement of atoms.

This provides an important clue: Lattice vibration is known to scatter electrons and cause resistance.

When a electron moves in a metal, it can attract the positive ions and bring them closer.



Another electron may then get attracted to the displaced ions.



http://hyperphysics.phy-astr.gsu.edu/hbase/solids/coop.html

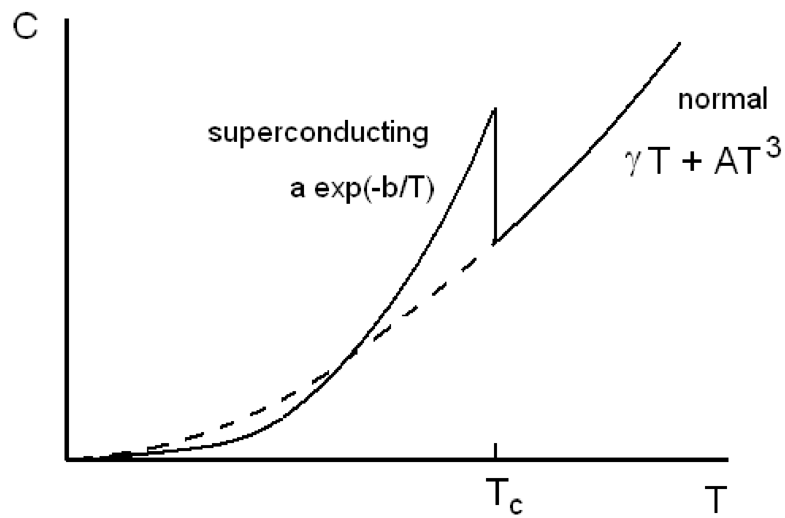
The attractive potential between electrons is much smaller than the kinetic energy of the two electrons. So it should not normally be able to bind the electrons together.

However, in this case, the two electrons are not in free space. They are in a Fermi sea - electrons stacked up to the Fermi energy.

In the 1950s, Leon Cooper showed that two electrons near the Fermi energy is is able to form a bound pair.

Bardeen, Cooper and Schrieffer (BCS) then developed a complete theory to that is able to explain the Meissner's effect, the zero resistance, the heat capacity behaviour, and other phenomena of superconductors.

As an example of a prediction by the BCS theory, recall the behaviour of heat capacity in a superconductor, $C = a\exp(-b/T)$.



This can be written in the form:

$$C_v = D\exp\left(-\frac{\Delta}{k_B T}\right)$$

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This looks like the Boltzmann factor, in which Δ is the energy between two levels. In the BCS theory, Δ is the energy needed to excite one electron from the BCS condensate.

This energy is now called the energy gap. It can be obtained directly from a heat capacity measurement by fitting the above formula.

BCS theory predicts that the energy gap and the transition temperature are related by:

$$2\Delta = 3.52 k_B T_c.$$

Rearranging the relation gives this ratio:

$$\frac{2\Delta}{k_B T_c} = 3.52.$$

The ratio for measured values are shown here:

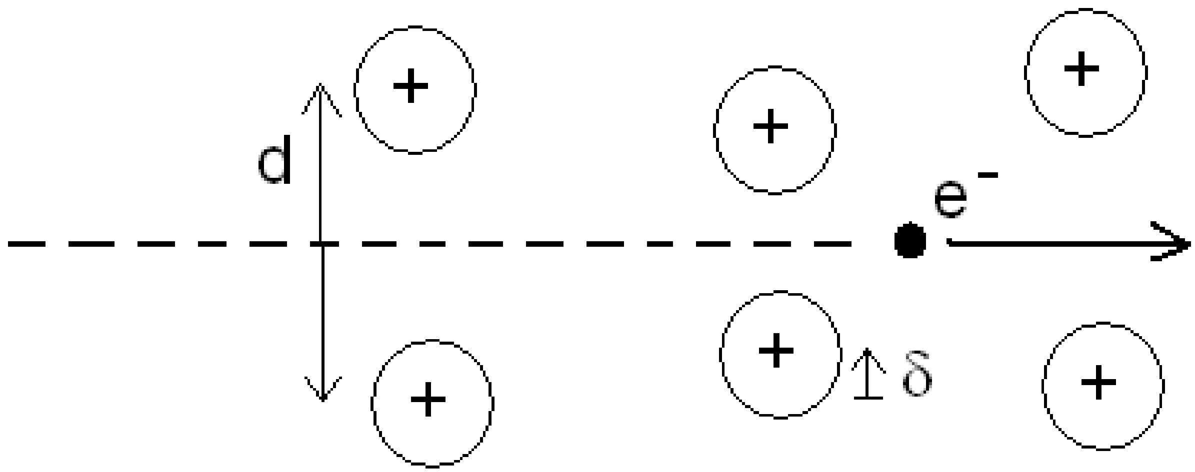
Superconductor	$2\Delta(0)/k_B T_c$
Al	3.37 ± 0.1
Cd	3.20 ± 0.1
Hg	4.60 ± 0.1
In	3.63 ± 0.1
Nb	3.84 ± 0.06
Pb	4.29 ± 0.04
Sn	3.46 ± 0.1
Ta	3.60 ± 0.1

Meservey and Schwarz, in Parks (1969) Superconductivity

The ratios are all fairly close to 3.52. This is one of many evidencethat supports the BCS theory.

More on Cooper pairs

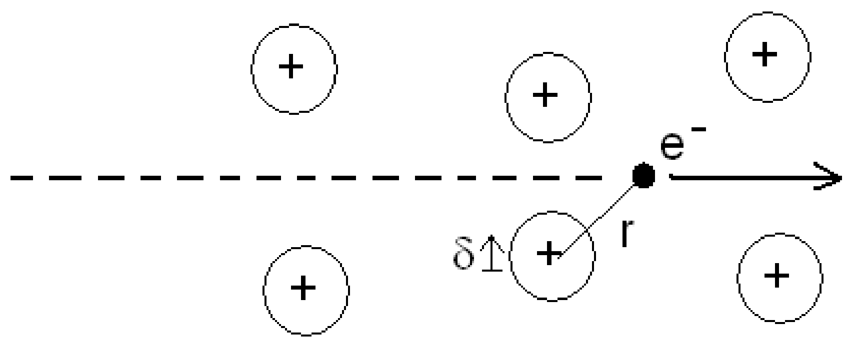
When an electron passes a positive ion, the ion experiences a force from the electron for a short time.



This impulse cause the ion to move and oscillate at the Debye frequency.

The average charge in a lattice is zero. The potential energy of an electron is due to an ion is

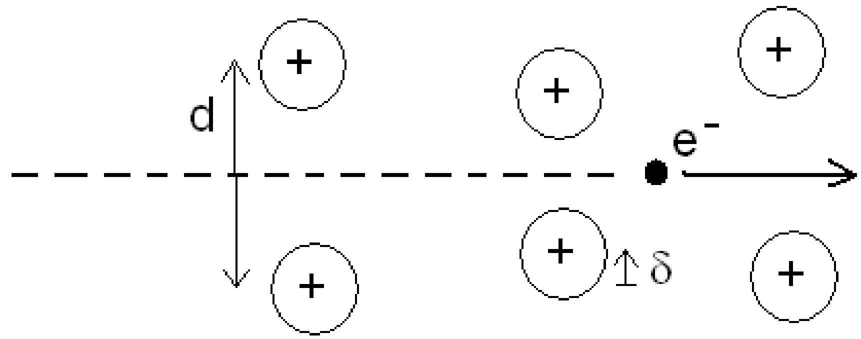
$$U = -\frac{e^2}{4\pi\epsilon_0 r}.$$



If the ion is displaced by an electron's attraction, the net potential is approximately the change in the ion's potential:

$$\delta U \approx \frac{dU}{dr}\delta r \approx -\frac{e^2}{4\pi\epsilon_0 r^2}\delta.$$

This net potential is only present near the ion. Further away, it will not be felt because other electrons would move around and cancel it (screening). So another electron would feel this potential only when r is about d, the distance between ions.



Let the displacement of the ion be δ. The new potential is

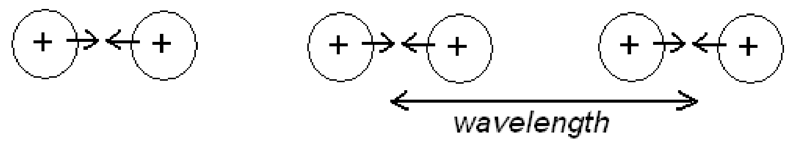
$$V = \delta U \approx -\frac{e^2}{4\pi\epsilon_0 d^2}\delta.$$

To estimate this, we need to find δ. For small displacement δ of the ion, the ion's motion is simple harmonic. We know then that: maximum velocity = amplitude x frequency. So when the ion receives a sudden attraction (impulse) from a passing electron, it takes off from rest with the velocity :

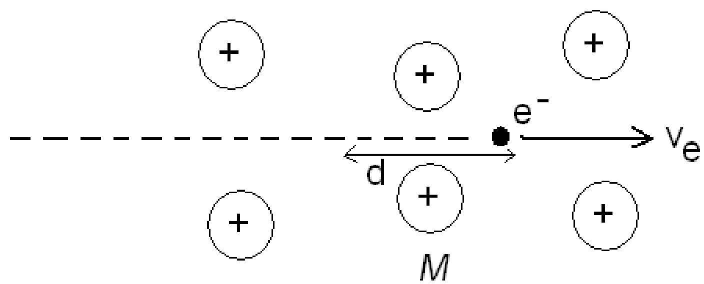
$$v_0 \approx \delta \times \omega_D.$$

To see that the Debye frequency ω_D is the correct frequency here, recall that ω_D is the maximum frequency in lattice vibration.

When a lattice vibrates the maximum frequency, the adjacent ions move in opposite phase, just as in the present case of two ions attracted by an electron passing in between.



To find v_0, we need to know the impulse (force x time) from the electron. The force is active only when the distance is within a distance of about d from the ion.



So the force and time, are respectively,

$$F \approx \frac{e^2}{4\pi\epsilon_0 d^2} \text{ and } \tau \approx \frac{d}{v_e},$$

where v_e is the velocity of the electron.

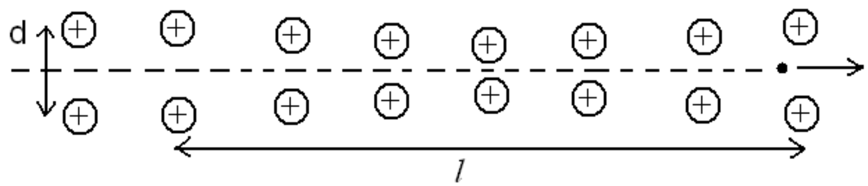
If the electron is at the Fermi level E_F, the velocity can be obtained from:

$$E_F = \frac{1}{2} m v_e^2.$$

Then the ion's velocity is

$$v_0 = \frac{F\tau}{M}$$

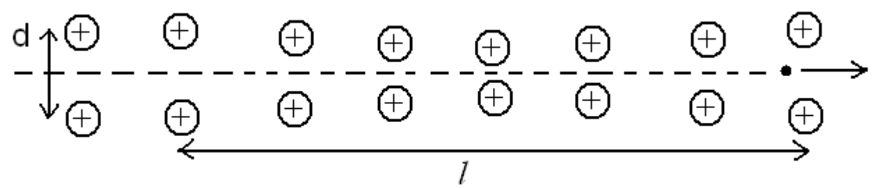
where M is the mass of the ion.



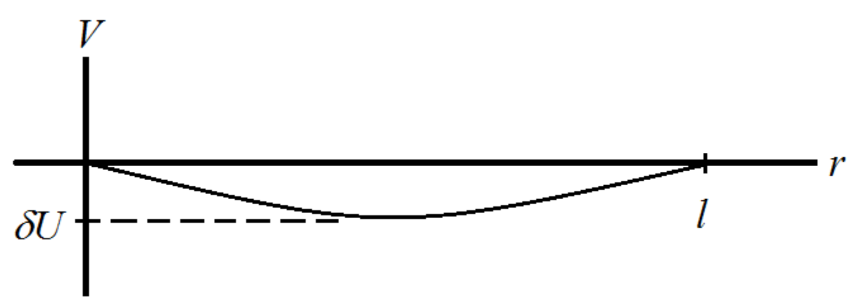
The attraction only exists in the narrow region between adjacent ions, and behind the passing electron. Also, it would only last until the displaced ion returns to its rest position. The time for this is the half of the period $2\pi/\omega_D$. So a passing electron with velocity v_e leaves behind a trail of displaced ions of length

$$l = v_e \times \frac{\pi}{\omega_D}.$$

This is also the length, or extent, of the attractive region. This attraction can only be felt by another electron travelling along nearly the same path, in the opposite direction. If they travel in the same direction, the electron in front would not feel the attraction from the electron behind.

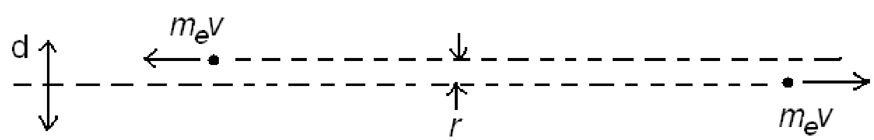


From the qualitative picture above of the displacements, we may sketch the potential well as follows.



where δU and l can be calculated from their formulae in earlier slides.

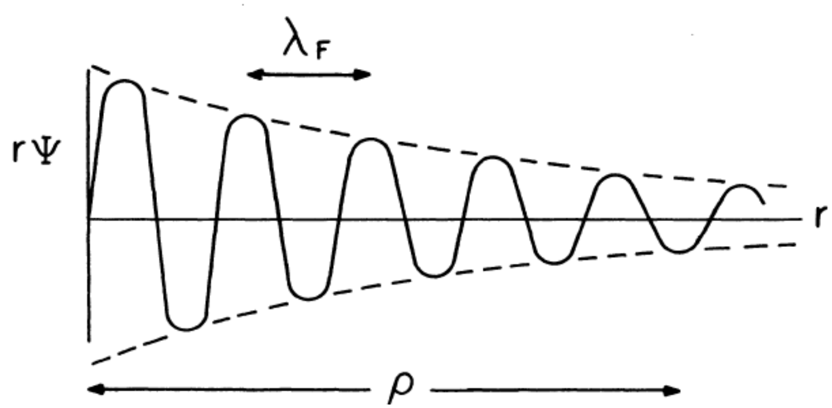
In order to know whether this attraction could lead to a bound state, we must solve the Schrodinger's equation to see if the wavefunction has a finite size (as opposed to a sine wave that goes on forever). Without going into the detailed solution, lets try and guess the shape of this wavefunction.



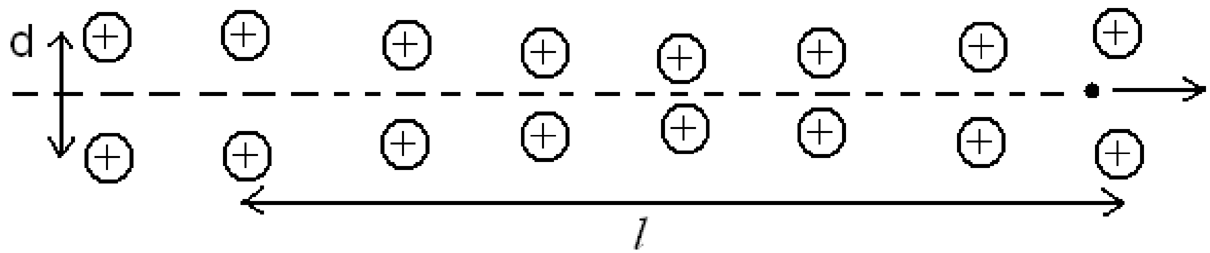
The attaction is very nearly along the line joining the two electrons, just like the Coulomb force. So the potential is effectively spherically symmetric. As the electrons are in oppostive directions (and the paths very close), we may assume that the angular momentum is zero.

The ground state wavefunction of the hydrogen atom also has spherical potential and zero angular momentum. So the wavefunction would be spherical. However, it would also have many oscillation, with the electrons have the large kinetic energy of E_F. The attractive potential is much weaker, and not normally enough to bind the electrons.

When Leon Cooper solved the Schrodinger's equation in 1956, he used a sum of sine waves for the wavefunction, with wavevectors k above the Fermi level. He showed that this can solve the equation. The solution is indeed a wavefunction with a finite size of about $300d$, 300 times the spacing between ions.



300 times the spacing between ions is close to the the length of the trail of displaced ions.



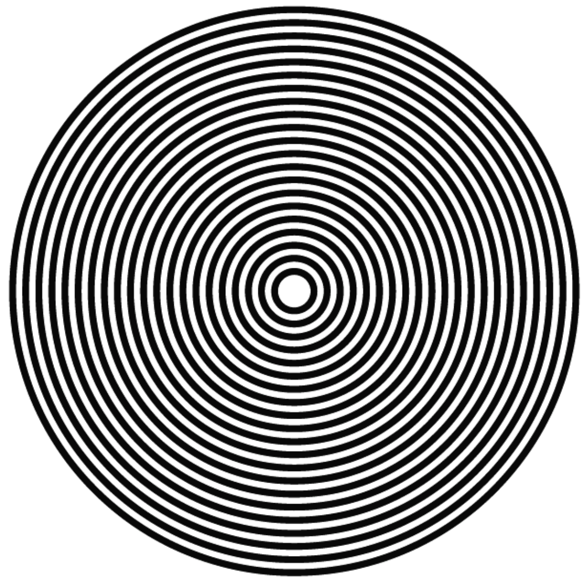
Using this formula

$$l = v_e \times \frac{\pi}{\omega_D}$$

and the formula for Debye frequency ω_D, the length of the trail can be calculated.

The length of this trail is also the maximum range of the attraction between the two electrons, so it is reasonable that this length is close to the size of the Cooper pair.

The wavefunction of an electron in a Cooper pair in real space is not unlike that of an electron around an atom.



Kadin, Spatial Structure of the Cooper Pair (2005)

There are many nodes because of the high kinetic energy, E_F (higher energy means shorter wavelength).

The size of the Cooper pair is a few hundred times the spacing between atoms, so there is a lot of overlap between Cooper pairs.

The most obvious property about a superconductor is the zero resistance. Unfortunately, there does not appear to be a simple way to explain this.

Cooper pairs can carry electric current, but why does it not get scattered by phonons and experience resistance?

Victor Weisskopf suggested that the Cooper pairs are packed like atoms in the helium-4 superfluid, and has zero resistance for similar reasons - rotons, excitations and Landau critical velocity.

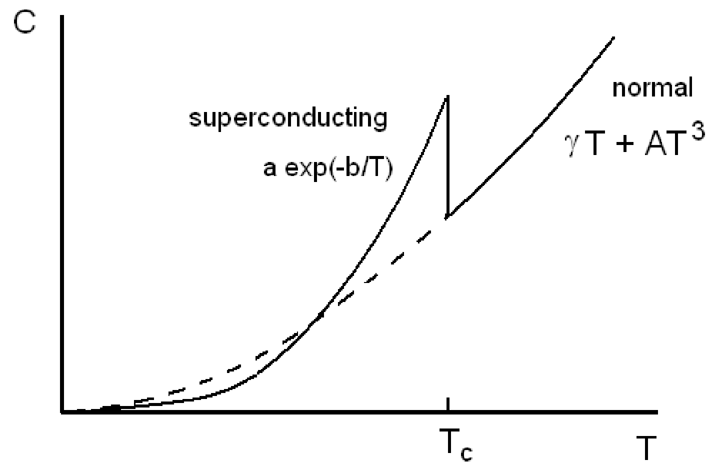
`http://cdsweb.cern.ch/record/880131/files/p1.pdf`

The Cooper pairs would flow like a superfluid, unless there is enough energy to break them. This would happen at the transition temperature, T_c.

Δ is the energy gap obtained by fitting Boltzmann distribution

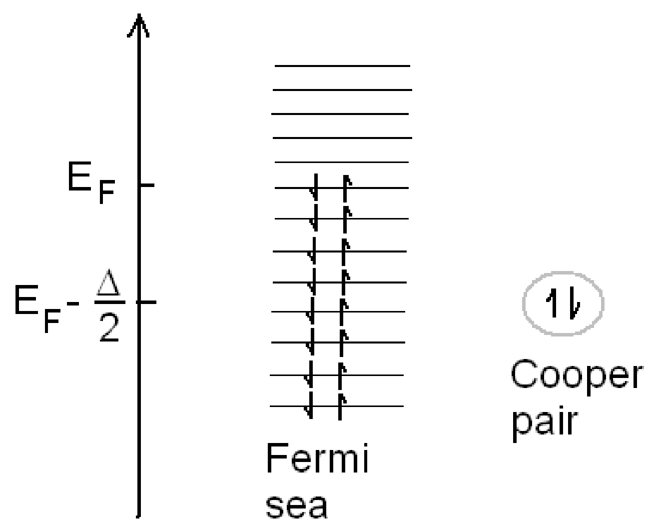
$$C_v = D \exp\left(-\frac{\Delta}{k_B T}\right)$$

to the heat capacity:



If we take Δ as the energy needed to break the Cooper pair, then we would expect that it is close to $k_B T_c$, because the Cooper pairs should start disappearing around T_c. This is consistent with measurements, which give $k_B T_c \approx \Delta$.

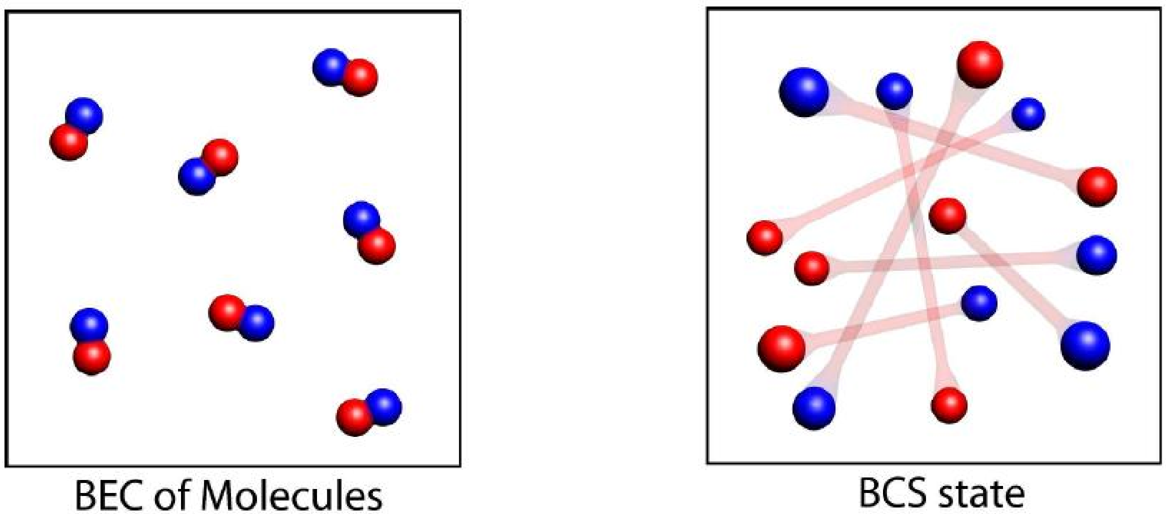
Two electrons at Fermi level have energy E_F each. If they form a Cooper pair, they give out energy Δ. So energy of the Cooper pair is $2E_F - \Delta$. The binding energy Δ is about $10^{-4}E_F$. If Δ is so much smaller, why do the electrons not come apart?



Physically, if a electron just leaves the Cooper pair, its wavefunction must change to that of a free particle, i.e. a sine function, with energy $E_F - \Delta/2$. However, this is just below E_F, where the states are fully occupied. This is not allowed by the exclusion principle.

The electrons in the pair have opposite spin, so that resultant spin of the Cooper pair is zero - it is a boson.

So, like the Bose-Einstein condensate, the Cooper pairs can condense into the ground state and form a condensate.



Ketterle and Zwierlein, Making, probing and understanding ultracold Fermi gases (2006)

However, because of the considerable overlap, the interaction is far more complex. It is normally called a BCS condensate instead.

Describe the behaviours a superconductor: zero resistance, Meissner's effect, and heat capacity. Explain the significance of the heat capacity behaviour.

Discuss why Faraday's law of electromagnetic induction cannot explain Meissner's effect. Explain Meissner's effect using a macroscopic wavefunction.

Explain the cause of London penetration depth. Derive a formula for it.

Explain the significance of the isotope effect. Suggest how electrons can attract each other.

Describe the effect of electron movement on positive ions in metal. Explain how the motion of the ions are related to Debye frequency.

Describe how electron movement leaves behind a trail of displaced ions and explain how to calculate the length of this trail. Suggest how this relates to the size of the Cooper pair.

Explain how the idea of superfluid is applied to the Cooper pair.

Worked Examples

Example 1

A 20 cm long aluminium cylinder has a radius of 2 cm. It is in a 0.001 T magnetic field that is parallel to its axis. The cylinder is superconducting, and the field penetrates 200 $\AA$ into the surface of the cylinder.

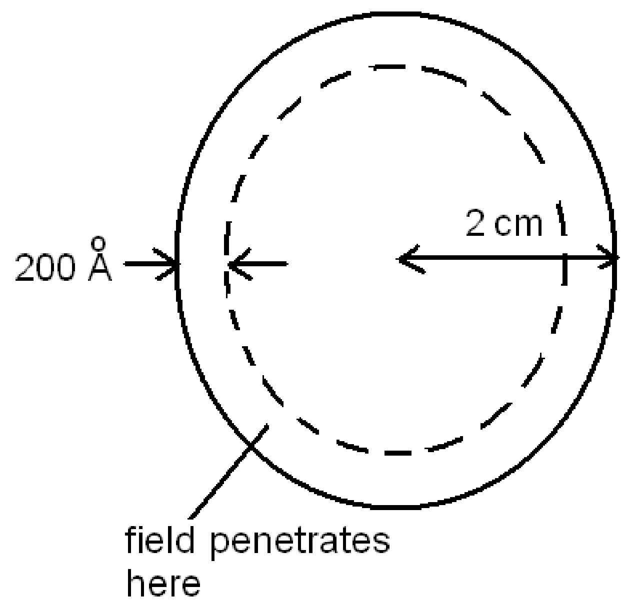
(i) Sketch the cross section of the cylinder. Indicate the area where the field has penetrated. Label the radius and the penetration depth.

(ii) Estimate this area.

(iii) Estimate the flux in the cylinder.

Solution

(i)



(ii) We can think of this as a long narrow strip with width 200 $\AA$ and length equal to circumference of the cross-section.

Then area = length x width = $2\pi \times (2\text{ cm}) \times 200\AA$

$= 2\pi\times(0.02\ \mathrm{m})\times 200\times 10^{-10} = 2.513\times 10^{-9}\ \mathrm{m}^2.$

(iii) Flux $=$ field x area $= 0.001$ T $\times 2.513\times 10^{-9}\ \mathrm{m}^2$

$= 2.513\times 10^{-12}$ T m^2.

Example 2

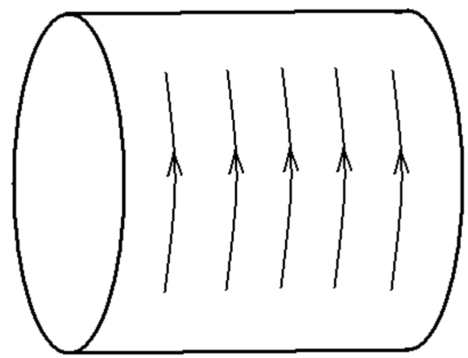
(i) With the help of a sketch, explain how the current circulating the cylinder in Example 1 is similar to a solenoid.

(ii) Write down the formula for the magnetic field in a solenoid, explaining the symbols used. What is meant by the total current circulating the solenoid and what is the formula?

(iii) The current circulating the cylinder exactly cancels the external field inside the cylinder. Use the solenoid formula to find the total current circulating the cylinder.

Solution

(i)



The current circulates the cylinder in a thin layer on the curved surface. It has the same effect as current carrying wires wound round the cylinder into a solenoid.

Because of this, the solenoid formula can also be used to relate the current on the cylindrical surface to the field inside the cylinder that this current produces.

(ii) Formula for the magnetic field in a solenoid is

$$B = \frac{\mu_0 N I}{L},$$

where L is the length of the solenoid, N is the number of turns and I is the current.

Total current is the sum of currents in all the turns. It is equal to NI.

(iii) Rearrange the solenoid formula to give the total current:

$$NI = \frac{BL}{\mu_0}.$$

The external field in Example 1 is 0.001 T. If the current exactly cancels this, then it must produce an opposing field of 0.001 T.

So $B = 0.001$ T. Therefore

$$NI = \frac{BL}{\mu_0} = 159.2 \text{ A}.$$

Example 3

This question follows from Examples 1 and 2.

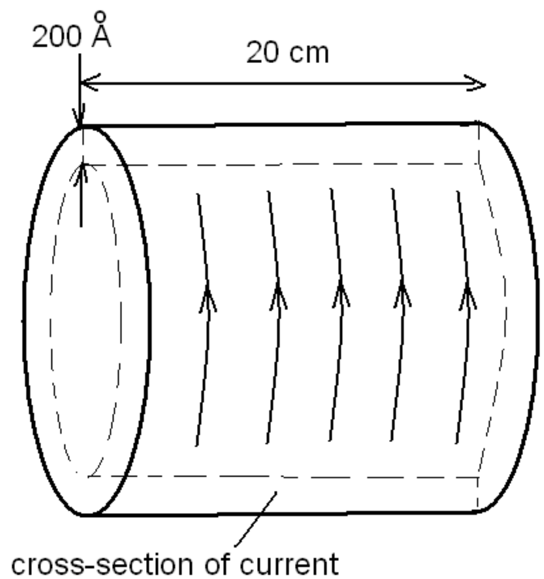
(i) The current circulates in a thin layer near the surface of the cylinder. Indicate a sketch of the cylinder a cross-section of this thin layer. Label the length and width of this cross-section.

(ii) Find the cross-sectional area of the current sheet on the cylinder in Example 1.

(iii) Find the current density in Example 1.

Solutions

(i)



(ii) The cross sectional area of the current sheet has a length of 20 cm and width of 200 $\mathring{A}$.

So the area $=$ 20 cm $\times 200\mathring{A} = 0.2$ m $\times 200 \times 10^{-10}$ m $=$ 4×10^{-9} m^2.

(iii) current density = current / area

$= 159.2$ A / $(4 \times 10^{-9}$ m$^2) = 3.979 \times 10^{10}$ A m^{-2}

Example 4

A piece of aluminium is at a temperature where it is a superconductor. When an electron travels through its lattice of positive ions, the electron leaves behind a trail of displaced ions.

(i) Write down the formula to estimate the natural frequency of the displaced ions, explaining the symbols used.

(ii) Estimate the natural frequency of the displaced ions. (Molar volume of aluminium is 10 cm $^{-3}$.. Speed of sound in aluminium is 6420 m/s.)

(iii) Estimate the time that a displaced ion takes to return to its rest position.

Solution

A piece of aluminium is at a temperature where it is a superconductor. When an electron travels through its lattice of positive ions, the electron leaves behind a trail of displaced ions.

(i) Formula to estimate the natural frequency of the displaced ions is Debye frequency:

$$\omega_D = \left(\frac{6N\pi^2 v^3}{V}\right)^{1/3}$$

where N is the number of atoms, v is the speed of sound in the solid, and V is the volume of the solid.

(ii)

$V = 10 \text{ cm}^3 = 10\times10^{-6} \text{ m}^3$

v = 6420 m/s

$N = N_A$

Substituting:

$$\omega_D = \left(\frac{6N\pi^2 v^3}{V}\right)^{1/3} = 9.807 \times 10^{13} \text{ rad/s.}$$

(iii) Estimate the time that a displaced ion takes to return to its rest position. It takes half a period:

$$\tau = \frac{\pi}{\omega_D} = 3.203 \times 10^{-14} \text{ s.}$$

Example 5

Following from Example 4:

(i) Write down the formula for Fermi energy. Calculate this energy. (Aluminium gives one conduction electron per atom.)

(ii) Calculate the velocity of an electron at Fermi level.

(iii) Estimate the length of the trail of displaced ions. Explain what this suggests about the size of the Cooper pair in aluminium.

Solutions

(i) Formula for Fermi energy is

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}.$$

$N = N_A$

$V = 10\ \text{cm}^3 = 10\times10^{-6}\ \text{m}^3$

$m = m_e$

Subtituting:

$$E_F = 8.896 \times 10^{-19}\ \text{J}.$$

(ii) At Fermi level, kinetic energy of electron = Fermi energy.

$$\frac{1}{2}m_e v^2 = E_F.$$

Velocity

$$v_e = \sqrt{\frac{2E_F}{m_e}}$$

substituting, we get

$$v_e = 1.398 \times 10^6 \text{ m/s}.$$

(iii)

length of the trail = distance electron travelled before a displaced ion return to rest position

So the length is

$$l = v_e\tau = 1.398 \times 10^6 \text{ m/s} \times 3.203 \times 10^{-14} \text{ s} = 4.477 \times 10^{-8} \text{ m}$$

This is the range of attraction between electrons in the Cooper pair. This suggests that the size of the Cooper pair is also about 4.477×10^{-8} m.