Cockcroft Institute Lectures

Special Relativity

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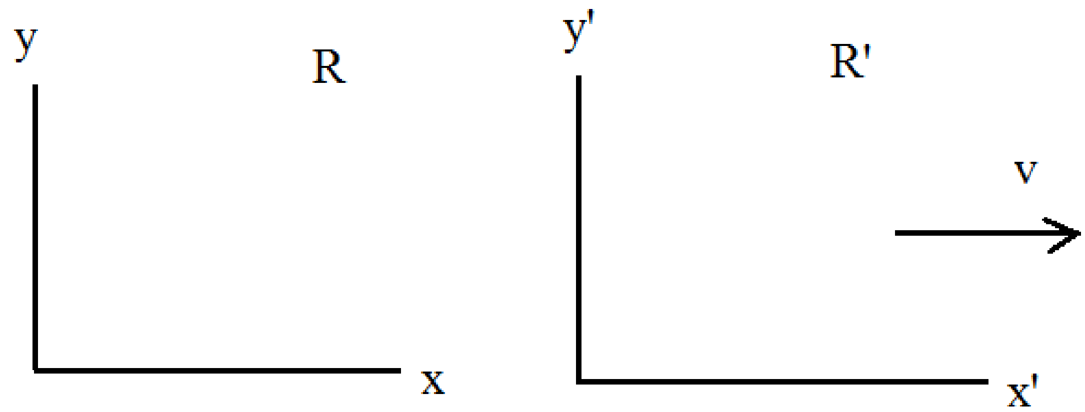
Autumn Term 2013

Objectives

We shall

- derive Lorentz transformation,
- derive $E = mc^2$,
- derive length contraction / time dilation,
- derive four-vectors,
- derive relativistic Hamiltonian, and
- work out a collider problem.



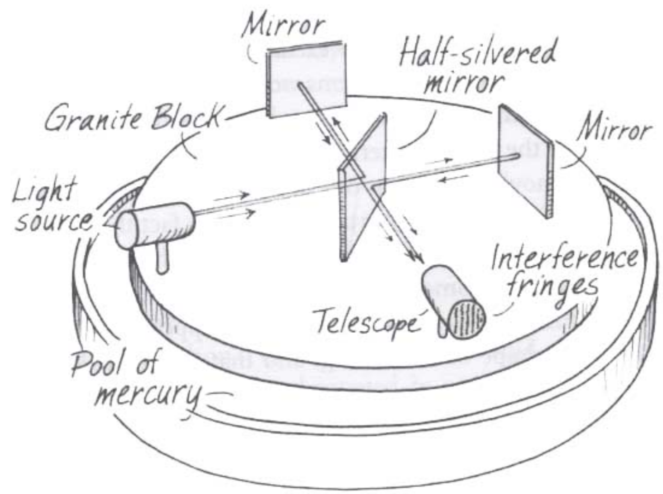
$$x' = x - vt$$
$$t' = t$$

- a linear transformation between (x, t) and (x', t').

Frame of reference essentially means viewpoint of observer.

Galilean transformation means that speed of light can change in another frame.

The Michelson-Morley experiment look for changes in intereference fringes as an interferometer was rotated.



http://sciencesummit.wordpress.com/2011/05/28/michelson%E2%80%93morley-experiment/

It would be sensitive to changes in light speed in different directions as Earth travels through space.

The null result suggests that speed of light might be same in different frames.

Postulate: Speed of light is same in all frames of reference.

This means that Galilean transformation must be wrong.

Need to find new one. Hope that it linear. Let

$$x' = \gamma x + bt \tag{1}$$
$$t' = Ax + Bt \tag{2}$$

Lets follow the origin of R', $x' = 0$.

To an observer in R, it is at some (x, t) and $x = vt$.

Substitute this into equation 1.

This gives

$$0 = \gamma vt + bt$$

and $b = -\gamma v$.

Substitute this into equation 1 gives

$$x' = \gamma(x - vt) \tag{3}$$

To an observer in R', velocity of R is -v. So

$$x = \gamma(x' + vt') \tag{4}$$

Since speed of light is the same in R and R', $t = x/c$ and $t' = x'/c$.

Substituting into equations 3 and 4 gives

$$x' = \gamma(1 - v/c)x$$
$$x = \gamma(1 + v/c)x'$$

Multiplying:

$$xx' = \gamma^2(1 - v^2/c^2)xx' \tag{5}$$

Solving gives the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Notice that this must be more than or equal to 1.

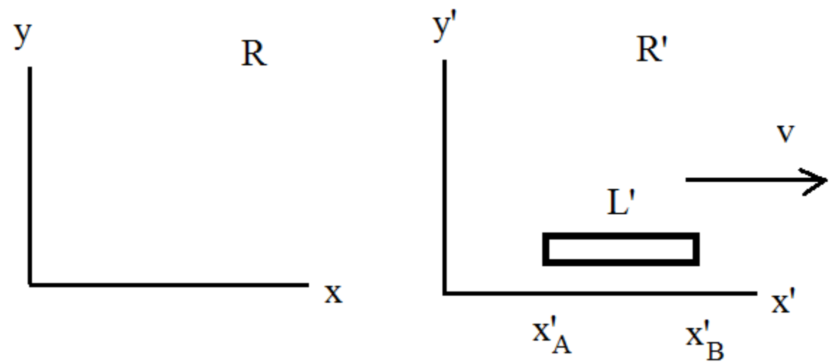
Substitute the light speed relations $x' = ct'$ and $t = x/c$ into equation 3 [$x' = \gamma(x - vt)$] gives

$$t' = \gamma(t - vx/c^2) \tag{6}$$

This completes the Lorentz transformation:

$$t' = \gamma(t - vx/c^2) \tag{7}$$
$$x' = \gamma(x - vt) \tag{8}$$
$$y' = y \tag{9}$$
$$z' = z \tag{10}$$

Suppose a ruler of length L' is at rest in frame R'. R' moves with velocity v relative to frame R. What is the length of the moving ruler when measure by an observer who is at rest in R?



Consider positions in R' of two ends of ruler. Relate to their positions in R:

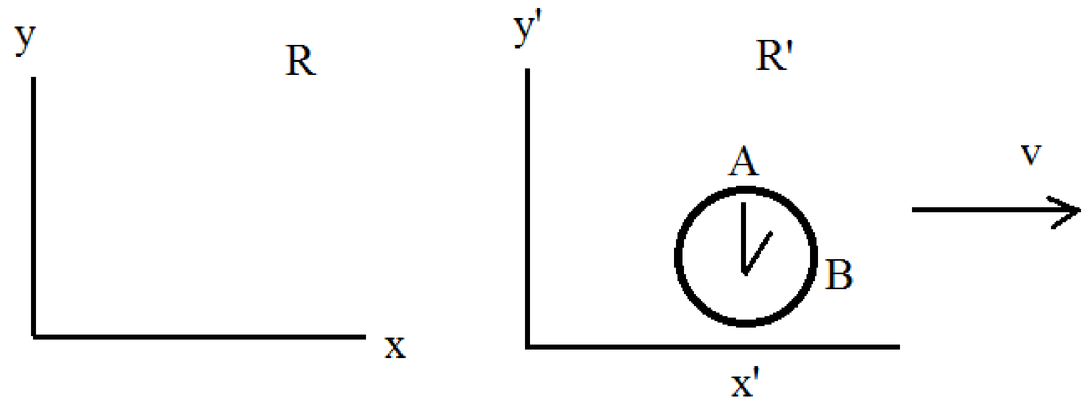
$$x'_A = \gamma(x_A - vt)$$
$$x'_B = \gamma(x_B - vt)$$

Subtracting:

$$L' = \gamma(x_B - x_A)$$

So in R, the length $= x_B - x_A = L'/\gamma$ is shorter because $\gamma > 1$. So the length has contracted.

A clock is at rest in frame R'. Two points on the clock are marked A and B. The clock measures the time taken for the minute hand to move from A to B. An observer at rest in frame R uses their own clock to measure this time.



Relate the times in R' to the times in R using Lorentz transformation:

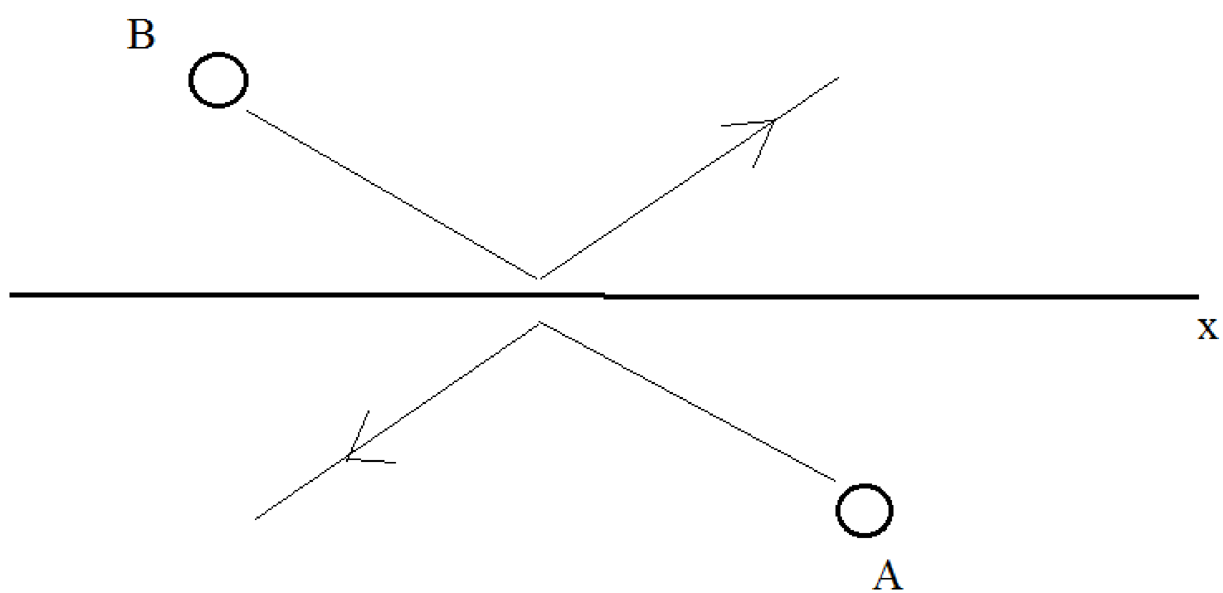
$$t_A = \gamma(t'_A + vx'/c^2)$$
$$t_B = \gamma(t'_B + vx'/c^2)$$

Subtracting, $t_B - t_A = \gamma(t'_B - t'_A)$. So duration in R' is shorter since $\gamma > 1$. Hence the time is dilated.

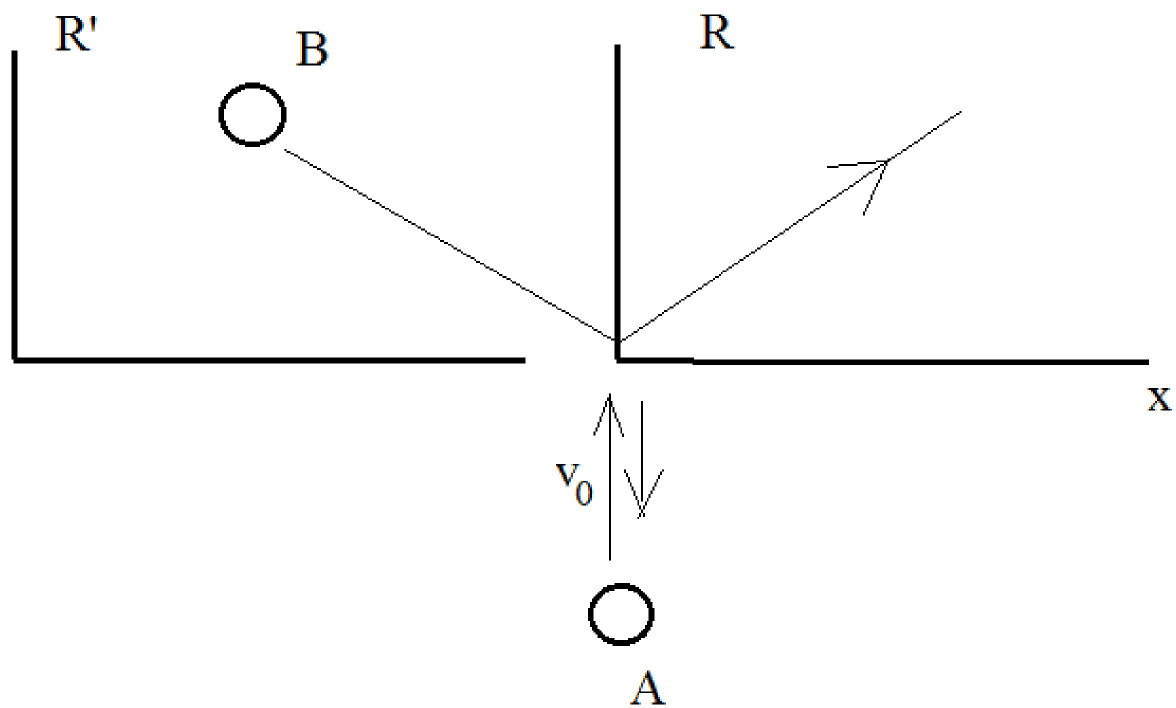
Under Lorentz transformation, momentum is not conserved.

To see this, consider two balls with speeds in equal and opposite directions.

Suppose that they bounce off each other at angle. Choose as x axis the line of symmetry of their paths:



Choose as frame R one in which A moves up and bounces down vertically.



Choose as frame R' one in which B moves down and bounces up vertically.

Because of time dilation, an observer at rest in R sees that the vertical velocity of ball B slows down to v_0/γ. The vertical momenta are:

Collision	A	B	Total
before	$+mv_0$	$-mv_0/\gamma$	$+mv_0(1 - 1/\gamma)$
after	$-mv_0$	$+mv_0/\gamma$	$-mv_0(1 - 1/\gamma)$

Total momenta before and after collision are not equal.

So momentum is not conserved!

If mass m increases to γm, the magnitudes of the vertical momenta become equal:

Collision	A	B	Total
before	$+mv_0$	$-\gamma m v_0/\gamma$	0
after	$-mv_0$	$+\gamma m v_0/\gamma$	0

Total momenta before and after collision are equal.

So momentum is now conserved.

So to conserve momentum, mass must increase with velocity as

$$\gamma m = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

However, as v approaches c, γm approaches infinity.

This suggests that no object with mass can travel faster than the speed of light.

Work done W by a force F on an object in free space is equal gain in kinetic energy E_K.

Newton's 2nd law, taking into account mass increase:

$$F = \frac{dp}{dt} = \frac{d\gamma mv}{dt}$$

The object starts from rest. So kinetic energy is

$$E_k = W = \int F dx = \int F \frac{dx}{dt} dt = \int \frac{dp}{dt} v dt = \int v dp$$

Integrating by parts,

$$
\begin{aligned}
E_k &= \int v\,dp \\
&= v'p\Big|_0^v - \int_0^v p\,dv' \\
&= \gamma m v^2 - \int_0^v \gamma m v'\,dv' \\
&= \gamma m v^2 - \int_0^v \frac{m v'}{\sqrt{1-\frac{v'^2}{c^2}}}dv' \\
&= \gamma m v^2 + mc^2\sqrt{1-\frac{v^2}{c^2}} - mc^2
\end{aligned}
$$

Simplifying:

$$E_k = \gamma mc^2 - mc^2$$

Since we assumed that γm is mass increase, maybe it is because the kinetic energy has mass.

By extension, maybe the mass of the body at rest can be converted to energy?

If so, we would expect that the resulting energy is

$$E = mc^2$$

because this part was subtracted off in the energy gain equation above.

Lots of maybe's when Einstein derived this, but since then confirmed by atomic bomb and nuclear energy.

Rewrite Lorentz transformation in matrix form:

$$\begin{pmatrix} ct_1' \\ x_1' \\ y_1' \\ z_1' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

where $\beta = v/c$.

Just for fun, take the transpose:

$$\begin{pmatrix} ct_2' & x_2' & y_2' & z_2' \end{pmatrix} = \begin{pmatrix} ct_2 & x_2 & y_2 & z_2 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

I have changed the subscripts from 1 to 2. That is ok because the transformation applies to any coordinate value.

While we are having fun, lets also reverse the signs of ct. It turns out that the equation is still correct if we also reverse the signs of $-\beta\gamma$:

$$\begin{pmatrix} -ct'_2 & x'_2 & y'_2 & z'_2 \end{pmatrix} = \begin{pmatrix} -ct_2 & x_2 & y_2 & z_2 \end{pmatrix} \begin{pmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

You can check that if you expand this, you will still get the correct Lorentz transformation equations.

Changing the signs of $-\beta\gamma$ gives the inverse matrix, since reversing velocity means reversing the transform.

See what happens if we now multiply this by the first equation on the last slide.

$$\begin{pmatrix} -ct'_2 & x'_2 & y'_2 & z'_2 \end{pmatrix} \begin{pmatrix} ct'_1 \\ x'_1 \\ y'_1 \\ z'_1 \end{pmatrix} =$$

$$\begin{pmatrix} -ct_2 & x_2 & y_2 & z_2 \end{pmatrix} \begin{pmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Since one matrix is inverse of the other, they cancel:

$$\begin{pmatrix} -ct'_2 & x'_2 & y'_2 & z'_2 \end{pmatrix} \begin{pmatrix} ct'_1 \\ x'_1 \\ y'_1 \\ z'_1 \end{pmatrix} = \begin{pmatrix} -ct_2 & x_2 & y_2 & z_2 \end{pmatrix} \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Each side is called an inner product.

$$\begin{pmatrix} -ct_2' & x_2' & y_2' & z_2' \end{pmatrix} \begin{pmatrix} ct_1' \\ x_1' \\ y_1' \\ z_1' \end{pmatrix} = \begin{pmatrix} -ct_2 & x_2 & y_2 & z_2 \end{pmatrix} \begin{pmatrix} ct_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

On the right is the inner product of two vectors in one frame.

On the left is the inner product after these vectors are Lorentz transformed to another frame.

The equation shows that inner products remain the same after any Lorentz transform. We say that inner products are invariant.

Any vector that transforms according to the Lorentz transformation is called a four-vector. To understand this better, lets make some more four-vectors.

We have seen one four-vector: 4-position $X = (ct, x, y, z)$.

Differentiate this with respect to proper time τ to give 4-velocity:

$$V = \frac{dX}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$$

Proper time is the time in the rest frame of the moving body. So it does not depend on Lorentz transformation. If we transform the above vector, we would just treat $d\tau$ as a constant factor.

The rest of the vector is (cdt, dx, dy, dz). This follows the same Lorentz transform as X since it is just the difference between two different (ct, x, y, z).

Therefore V obeys the Lorentz transformation and is a four-vector.

To simplify, use familiar 3D notations

$$\mathbf{x} = (x, y, z)$$
$$\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

These familiar vectors we call 3-vectors.

Also, time dilation means that proper time runs slower since the body moves relative to other frames. So $dt > d\tau$. So

$$\frac{dt}{d\tau} = \gamma$$

So

$$X = (ct, \mathbf{x})$$
$$V = \frac{dX}{d\tau} = (c\gamma, \gamma\mathbf{v})$$

In this way, we can make new 4-vectors just by differentiating those that we know:

4-velocity $V = \dfrac{dX}{d\tau} = \gamma(c, \mathbf{v})$

4-momentum $P = mV = m_0\gamma(c, \mathbf{v}) = (mc, \mathbf{p}) = \left(\dfrac{E}{c}, \mathbf{p}\right)$

4-force $F = \dfrac{dP}{d\tau} = \gamma\dfrac{dP}{dt} = \gamma\left(c\dfrac{dm}{dt}, \dfrac{d\mathbf{p}}{dt}\right)$

So the inner product of any two of these 4-vectors is invariant (with respect to Lorentz transformation of the 4-vectors).

By extension, we call the familiar 3D vectors 3-vectors:

3-velocity $\mathbf{v} = (v_x, v_y, v_z)$

3-momentum $\mathbf{p} = (p_x, p_y, p_z)$

3-force $\mathbf{f} = \dfrac{d\mathbf{p}}{dt} = m_0 \dfrac{d\gamma\mathbf{v}}{dt}$

where the last one is the modified form of Newton's second law.

These formulae are often used:

$$
\begin{aligned}
&\text{relative velocity } \beta = \frac{v}{c} \\
&\text{Lorentz factor } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
&\text{momentum } p = mv = m_0\gamma\beta c \\
&\text{kinetic energy } T = (m - m_0)c^2 = m_0c^2(\gamma - 1) \\
&\text{total energy } E^2 = p^2c^2 + m_0^2c^4
\end{aligned}
\tag{11}
$$

The last one

$$E^2 = p^2c^2 + m_0^2c^4$$

is obtained using the invariance of 4-momentum P with itself.

$$P = \left(\frac{E}{c}, \mathbf{p}\right)$$

So inner product with itself is $-E^2/c^2 + p^2$.

In the rest frame of the body, $P = (m_0c^2/c, 0)$. So the inner product is $m_0^2c^2$.

So $-E^2/c^2 + p^2 = m_0^2c^2$, leading to the total energy equation above.

Classical mechanics generalises Newtonian mechanics into a method for systems with many coordinates. This method is called Hamiltonian mechanics. It is widely used in accelerator physics for beam dynamics calculations.

1. Define a function $H = T + V$. T is kinetic energy, V is potential energy so H is total energy. H is a function of momentum p_i and position q_i, where i means different coordinates.

2. The equation of motion is obtained using the Hamiltonian equations:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

As example, consider a mass m on a spring k.

Kinetic energy $T = \frac{p^2}{2m}$

Potential energy $V = \frac{1}{2}kx^2$

So Hamiltonian $H = T + V = \frac{p^2}{2m} + \frac{1}{2}kx^2$

The coordinates are momentum $p_1 = p$ and position $q_1 = x$.

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} = -kx$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

The first equation gives the spring equation $f = -kx$ which can be also be derived from Newton's 2nd law.

The second one gives the familiar $v = p/m$.

An object of mass m_0 moves in a potential field $V(x)$.

Kinetic energy

$$T = \gamma m_0 c^2 - m_0 c^2$$

So Hamiltonian

$$H = T + V = \gamma m_0 c^2 - m_0 c^2 + V$$

Later, when we differentiate, the constant term $m_0 c^2$ will give 0. So we can leave it out:

$$H = \gamma m_0 c^2 + V$$

We need H to be a function of p, so use this formula for energy:

$$H = \sqrt{p^2 c^2 + m_0^2 c^4} + V$$

To check that this is correct, must use Hamilton's equations to see if we can get back Newton's equations:

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{cp}{\sqrt{p^2 + m_0^2 c^2}}$$

First equations is Newton's 2nd law, since negative potential gradient on right side is force.

2nd equation has v on left side. Can be rearranged to

$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0 v$$

So both Hamilton's equations are correct.

We want to collide a proton p_1 with anti-proton p_2 to produce W particles W_1 and W_2. Mass of a W particle is $M_0 = 100m_0$, where m_0 is proton mass.

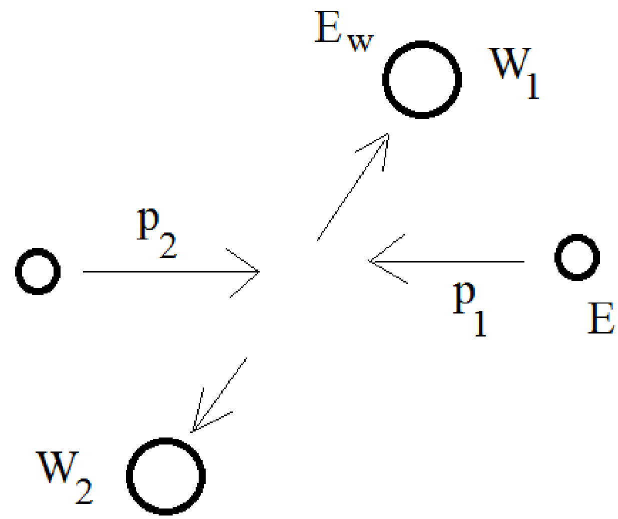
Two ways:

Experiment 1: p_1 and p_2 have equal and opposite velocities.

Experiment 2: p_1 is at rest.

Compare the energies needed.

Resultant momentum is zero, so centre of mass (CoM) is at rest - centre of mass frame.

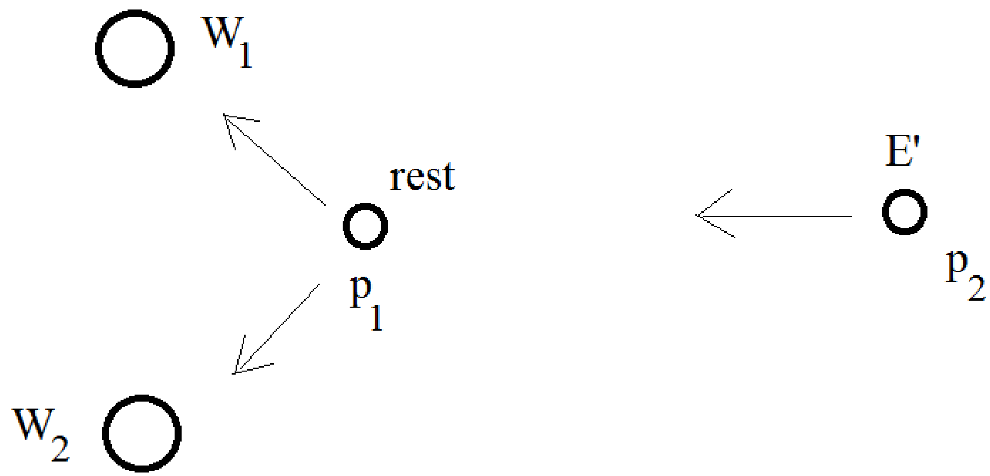


4-momenta before collision:

Collision		
before	p_1 $(E/c, \mathbf{p})$	p_2 $(E/c, -\mathbf{p})$

Energy conservation means that $E = E_W$. So E_W must be $> M_0c^2$ to produce a W. So at least $2 \times 100m_0c^2$ is needed in this experiment.

One proton is at rest. We call it the fixed target.



4-momenta before collision:

Collision before	p_1 $(m_0c, 0)$	p_2 $(E'/c, \mathbf{p}')$

Copy here the momenta before collision in the 2 frames:

Frames	p_1	p_2	$p_1 + p_2$
fixed target	$(m_0 c, 0)$	$(E'/c, \mathbf{p}')$	$(m_0 c + E'/c, \mathbf{p}')$
centre of mass	$(E/c, \mathbf{p})$	$(E/c, -\mathbf{p})$	$(2E/c, 0)$

Try inner product between p_1 and $p_1 + p_2$. (This happens to work $\ddot\smile$).

inner product in fixed target frame = inner product in centre of mass frame

$$-m_0 c \times \left(\frac{E'}{c} + m_0 c\right) + 0 \times \mathbf{p}' = -\frac{E}{c} \times \frac{2E}{c} + 0 \times \mathbf{p}$$

Simplifying gives

$$E' + m_0c^2 = \frac{2E^2}{m_0c^2}$$

From CoM case, we know that $E > 100m_0c^2$. So

$$E' + m_0c^2 = \frac{2E^2}{m_0c^2} > \frac{2(100m_0c^2)^2}{m_0c^2}$$

Rearranging gives

$$E' > (20000 - 1)m_0c^2$$

So the fixed target Expt 2 needs about 200 times more energy than the centre of mass Expt 1.

In Conclusion

We have

- derived Lorentz transformation,
- derived $E = mc^2$,
- derived length contraction / time dilation,
- derived four-vectors,
- derived relativistic Hamiltonian, and
- worked out a collider problem.